Macdonald symmetric functions for partitions with complex parts

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ABSTRACT. Using a recently discovered recursion formula for Macdonald polynomials, we propose an extension of the latter to Macdonald symmetric functions indexed by partitions with complex parts. These appear to possess nice properties, including a closed form evaluation formula. Our analysis involves summation theorems for (multivariable) basic hypergeometric series.

RÉSUMÉ. En utilisant une formule de récurrence découverte récemment pour les polynômes de Macdonald, on propose une extension des derniers aux fonctions symétriques de Mcdonald indexées par des partitions avec des parts complexes. Ces extensions semblent posséder de propriétés remarquables, incluant une forme close d'évaluation. Notre étude fait intervenir des théorèmes de sommation pour des séries hypergéométriques (multivariables) basiques.

1. Introduction

A_{n-1} *Macdonald polynomials* (or *Macdonald polynomials of type* A, or simply *Macdonald polynomials*) are a family of symmetric multivariable orthogonal polynomials associated with the irreducible reduced root system A_{n-1}. They were introduced by I. G. Macdonald [**17**] in the 1980's. See [**18**, Ch. 6] for a standard reference.

Macdonald polynomials (of type A) are indexed by integer partitions, and form a basis of the algebra of symmetric functions with rational coefficients in two parameters q and t. They generalize many classical bases of this algebra, including monomial, elementary, Schur, Hall–Littlewood, and Jack symmetric functions. These particular cases correspond to various specializations of the indeterminates q and t. In terms of basic hypergeometric series, the Macdonald polynomials correspond to a multivariable generalization of the *continuous* q*-ultraspherical polynomials*, see [**12**].

A principal tool for studying q-orthogonal polynomials (see e.g. [**8**]) is the theory of basic hypergeometric series (cf. [**6**]), rich of identities, having applications in different areas such as combinatorics, number theory, statistics, and physics (cf. [**1**]). Hypergeometric and basic hypergeometric series undoubtedly play a prominent role in special functions, see [**2**]. Even in one variable, they are still an object of active research. A notable recent advance includes *elliptic* (or *modular*) *hypergeometric series* (surveyed in [**6**, Ch. 11] and [**23**]) which is a one-parameter generalization of basic hypergeometric series, first introduced by Frenkel and Turaev [**5**] in a study related to statistical mechanics.

In joint work with Lassalle, the present author successfully inverted the Pieri formula for Macdonald polynomials. The result (which was subsequently also derived by Lassalle [**15**] by alternative means, namely by functional equations) was a new recursion formula for the Macdonald polynomials Q_λ, the recursion being on the row length of the indexing partition λ, see [**16**, Th. 4.1]. More precisely, the formula expands a Macdonald polynomial of row length $n+1$ into products of one row and n row Macdonald polynomials with explicitly determined coefficients (being rational functions in q and t). The recursion formula is an

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$(n+1)$-variable extension of Jing and Józefiak's [**9**] well-known two-variable result and, as such, represents the Macdonald polynomial extension of the celebrated Jacobi–Trudi identity for Schur functions [**18**, p. 41, Eq. (3.4)].

In [**22**] it is shown that the Pieri formula and the recursion formula both respresent two different multivariable generalizations of the terminating very-well-poised ${}_6\phi_5$ summation. More precisely, this interpretation comes from the explicit structure the two formulae have after application of (the analytic continuation of) principal specialization to both sides of the respective identities.

Motivated by the basic hypergeometric analysis, we use the recursion formula of [**16**, Th. 4.1] to define A_{n-1} Macdonald polynomials for "complex partitions", no longer indexed by sequences of non-increasing nonnegative integers but by arbitrary finite sequences of complex numbers. These A_{n-1} Macdonald symmetric functions are identified as certain multivariable nonterminating basic hypergeometric series which appear to satisfy some nice properties (among which is a nice closed form evaluation formula that extends the usual one in the polynomial special case).

This extended abstract is organized as follows: In Section 2 we recollect some items we need on basic hypergeometric series in one and several variables, hereby listing several summation formulae explicitly. In Section 3 we review some facts on A_{n-1} Macdonald polynomials and, in particular, state the Pieri and the recursion formula. Finally, in Section 4 we introduce Macdonald symmetric functions indexed by partitions with complex parts.

2. (Multivariable) basic hypergeometric series

First we recall some standard notations for q-series and basic hypergeometric series (see Gasper and Rahman's text [**6**], for a standard reference). In the following we shall consider q a (fixed) complex parameter with $0<|q|<1$.

For a complex parameter a, we define the q-shifted factorial as

$$(a;q)_\infty=\prod_{j\geq 0}(1-aq^j), \qquad \text{and} \qquad (a;q)_k=\frac{(a;q)_\infty}{(aq^k;q)_\infty}, \tag{2.1}$$

for arbitrary k (can be any complex number). We shall also occasionally use the condensed notation

$$(a_1,\dots,a_m;q)_k=(a_1;q)_k\dots(a_m;q)_k,$$

for arbitrary k.

The basic hypergeometric ${}_r\phi_{r-1}$ series with upper parameters $a_1,\dots,a_r$, lower parameters $b_1,\dots,b_{r-1}$, base q, and argument z is defined as follows:

$${}_r\phi_{r-1}\left[\begin{matrix}a_1,a_2,\dots,a_r\\ b_1,b_2,\dots,b_{r-1}\end{matrix};q,z\right]=\sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_k}{(q,b_1,\dots,b_{r-1};q)_k}\,z^k.$$

An ${}_r\phi_{r-1}$ series terminates if one of its upper parameters is of the form q^{-m} with $m=0,1,2,\dots$, because

$$(q^{-m};q)_k=0, \qquad k=m+1,m+2,\dots.$$

The ${}_r\phi_{r-1}$ series, if it does not terminate, converges absolutely for $|z|<1$.

In our computations throughout this paper, we make frequent use of some elementary identities for q-shifted factorials, listed in [**6**, Appendix I]. We list a few important summation theorems from [**6**, Appendix II], for quick reference.

We start with the (nonterminating) q-binomial theorem (cf. [**6**, Eq. (II.2)])

$${}_1\phi_0\left[\begin{matrix}a\\ -\end{matrix};q,z\right]=\frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z|<1, \tag{2.2}$$

an identity first discovered by Cauchy [**3**] in 1843.

Further, we have the q-Gauß summation (cf. [**6**, Eq. (II.8)]),

$${}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix};q,\frac{c}{ab}\right]=\frac{(c/a,c/b;q)_\infty}{(c,c/ab;q)_\infty}, \qquad |c/ab|<1, \tag{2.3}$$

which is due to Heine [**7**].

An identity of fundamental importance is Rogers' [**21**] nonterminating ${}_6\phi_5$ summation theorem (cf. [**6**, Eq. (II.20)])

$$ {}_6\phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d \end{matrix}; q, \frac{aq}{bcd}\right] = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}, \qquad |aq/bcd| < 1. \tag{2.4}$$

We will need the following identity from [**20**, Th. 7.6] which generalizes the q-Gauß summation in (2.3):

Proposition 2.1 ((Milne) An A_{n-1} nonterminating ${}_2\phi_1$ summation). *Let $a_1, \dots, a_n$, b, c, and $u_1, \dots, u_n$ be indeterminate. Then*

$$\sum_{k_1,\dots,k_n \ge 0} \prod_{1\le i<j\le n} \frac{u_i q^{k_i} - u_j q^{k_j}}{u_i - u_j} \prod_{i,j=1}^n \frac{(a_j u_i/u_j; q)_{k_i}}{(q u_i/u_j; q)_{k_i}} \cdot \frac{(b; q)_{|\mathbf{k}|}}{(c; q)_{|\mathbf{k}|}} \left(\frac{c}{a_1 \dots a_n b}\right)^{|\mathbf{k}|} = \frac{(c/a_1 \dots a_n, c/b; q)_\infty}{(c, c/a_1 \dots a_n b; q)_\infty},$$

provided $|c/a_1 \dots a_n b| < 1$.

We also need the following recent identity from [**22**, Cor. 4.3] which generalizes Rogers' nonterminating ${}_6\phi_5$ summation in (2.4):

Proposition 2.2 (A multivariable nonterminating ${}_6\phi_5$ summation). *Let b, d, $t_0, t_1, \dots, t_n$, and $u_1, \dots, u_n$ be indeterminate. Then*

$$\begin{aligned}
&\sum_{k_1,\dots,k_n \ge 0} \prod_{i,j=1}^n \frac{(q u_i/t_i u_j; q)_{k_i}}{(q u_i/u_j; q)_{k_i}} \prod_{1\le i<j\le n} \frac{(t_j u_i/u_j; q)_{k_i - k_j}}{(q u_i/t_i u_j; q)_{k_i - k_j}} \prod_{1\le i<j\le n} \frac{1}{u_i - u_j} \\
&\quad \times \det_{1\le i,j\le n}\left[(u_i q^{k_i})^{n-j}\left(1 - t_i^{j-n-1} \frac{1 - t_0 u_i q^{k_i}}{1 - t_0 u_i q^{k_i}/t_i} \prod_{s=1}^n \frac{u_i q^{k_i} - u_s}{u_i q^{k_i}/t_i - u_s}\right)\right] q^{\sum_{i=1}^n (1-i)k_i} \\
&\quad \times \prod_{i=1}^n \frac{(t_0 u_i q/t_i, b u_i; q)_{k_i}}{(t_0 u_i q, t_0 u_i q/d t_i; q)_{k_i}} t_i^{i k_i - \sum_{j=1}^i k_j} \cdot \frac{(d; q)_{|\mathbf{k}|}}{(t_0 q/b t_1 \cdots t_n; q)_{|\mathbf{k}|}} \left(\frac{t_0 q}{bd}\right)^{|\mathbf{k}|} \\
&\qquad = \frac{(t_0 q/b, t_0 q/b d t_1 \cdots t_n; q)_\infty}{(t_0 q/bd, t_0 q/b t_1 \cdots t_n; q)_\infty} \prod_{i=1}^n \frac{(t_0 u_i q/t_i, t_0 u_i q/d; q)_\infty}{(t_0 u_i q, t_0 u_i q/d t_i; q)_\infty},
\end{aligned}$$

provided $|t_0 q/bd| < 1$.

3. Preliminaries on A_{n-1} Macdonald polynomials

Standard references for Macdonald polynomials are [**17**], [**18**, Ch. 6], and [**19**]. In the following, we recollect some facts we need.

Let $X = \{x_1, x_2, x_3, \dots\}$ be an infinite set of indeterminates and $\mathcal{S}$ the algebra of symmetric functions of X with coefficients in $\mathbb{Q}$. There are several standard algebraic bases of $\mathcal{S}$. Among these there are the *power sum* symmetric functions, defined by $p_k(X) = \sum_{i\ge1} x_i^k$, for integer $k \ge 1$. Two other standard algebraic bases are the *elementary* and *complete* symmetric functions $e_k(X)$ and $h_k(X)$, which are defined, for integer $k \ge 0$, by their generating functions

$$\prod_{i\ge1}(1 + u x_i) = \sum_{k\ge0} u^k\, e_k(X), \qquad \prod_{i\ge1} \frac{1}{1 - u x_i} = \sum_{k\ge0} u^k\, h_k(X).$$

A partition $\lambda = (\lambda_1, \lambda_2, ...)$ is a weakly decreasing (finite or infinite) sequence of nonnegative integers, with a finite number of positive integers, called parts. The number of positive parts is called the length of λ and denoted $l(\lambda)$. If $l(\lambda) = n$, we often suppress any zeros appearing in the (sequential) representation of λ and write $\lambda = (\lambda_1, \dots, \lambda_n)$. For any integer $i \ge 1$, $m_i(\lambda) = \mathrm{card}\{j : \lambda_j = i\}$ is the *multiplicity* of i in λ. Clearly $l(\lambda) = \sum_{i\ge1} m_i(\lambda)$. We shall also write $\lambda = (1^{m_1}, 2^{m_2}, 3^{m_3}, \dots)$ (where the parts now appear in increasing order). We set

$$z_\lambda = \prod_{i\ge1} i^{m_i(\lambda)} m_i(\lambda)!.$$

For any partition λ, the symmetric functions e_λ, h_λ and p_λ are defined by

$$f_\lambda = \prod_{i=1}^{l(\lambda)} f_{\lambda_i} = \prod_{i\ge1} (f_i)^{m_i(\lambda)}, \tag{3.1}$$

where f_i stands for e_i, h_i, or p_i, respectively. (Here and in the following we sometimes omit writing out the argument X of the function, for brevity, assuming there is no confusion.) These e_λ, h_λ, and p_λ now each form a linear (vector space) basis of $\mathcal{S}$. Another classical basis is formed by the monomial symmetric functions m_λ, defined as the sum of all distinct monomials whose exponent is a permutation of λ.

Let $\mathbb{Q}(q,t)$ be the algebra of rational functions in the two indeterminates q,t, and $\mathsf{Sym} = \mathcal{S}\otimes\mathbb{Q}(q,t)$ the algebra of symmetric functions with coefficients in $\mathbb{Q}(q,t)$.

For any $k\ge 0$, the *modified complete* symmetric function $g_k(X;q,t)$ is defined by the generating function

$$\prod_{i\ge 1}\frac{(tux_i;q)_\infty}{(ux_i;q)_\infty}=\sum_{k\ge 0}u^k g_k(X;q,t). \tag{3.2}$$

It is often written in plethystic notation [**14**, p. 223], that is

$$g_k(X;q,t)=h_k\left[\frac{1-t}{1-q}\,X\right].$$

The symmetric functions $g_k(q,t)$ form an algebraic basis of Sym. Their explicit expansion in terms of power sums and monomial symmetric functions has been given by Macdonald [**18**, pp. 311 and 314] and in terms of other classical bases by Lassalle [**14**, Sec. 10, p. 237]. The functions $g_\lambda(q,t)$, defined as in (3.1) and (3.2), form a linear basis of Sym.

We are ready to define the Macdonald polynomials $P_\lambda(X;q,t)$. On one hand, they are of the form (recalling that m_λ denotes a monomial symmetric function)

$$P_\lambda(q,t)=m_\lambda+\text{a linear combination of the } m_\mu \text{ for } \mu \text{ preceding } \lambda \text{ in lexicographical order.}$$

Furthermore, they form an orthogonal basis of Sym with respect to the scalar product $<,>_{q,t}$ defined by

$$<p_\lambda,p_\mu>_{q,t}\;=\delta_{\lambda\mu}\,z_\lambda\prod_{i=1}^{l(\lambda)}\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Although these two conditions already *overdetermine* the symmetric functions $P_\lambda(X;q,t)$, the latter can be shown to exist (and are moreover unique), see [**18**, p. 322].

Let $Q_\lambda(q,t)$ denote the dual basis of $P_\lambda(q,t)$ for this scalar product. These are also called Macdonald polynomials, they differ from the latter only by a rational function of q and t. More precisely, one has

$$Q_\lambda(X;q,t)=b_\lambda(q,t)\,P_\lambda(X;q,t), \tag{3.3}$$

with $b_\lambda(q,t)={<P_\lambda(q,t),P_\lambda(q,t)>_{q,t}}^{-1}$ specified as follows (see [**18**, p. 339, Eq. (6.19)] and [**11**, Prop. 3.2]):

$$\begin{aligned} b_\lambda(q,t)&=\prod_{1\le i\le j\le l(\lambda)}\frac{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_{\lambda_j-\lambda_{j+1}}}{(q^{\lambda_i-\lambda_j+1}t^{j-i};q)_{\lambda_j-\lambda_{j+1}}}=\prod_{1\le i\le j\le l(\lambda)}\frac{(qt^{j-i};q)_{\lambda_i-\lambda_j}\,(t^{j-i+1};q)_{\lambda_i-\lambda_{j+1}}}{(t^{j-i+1};q)_{\lambda_i-\lambda_j}\,(qt^{j-i};q)_{\lambda_i-\lambda_{j+1}}}\\ &=\prod_{i=1}^{n}\frac{(t^{n+1-i};q)_{\lambda_i}}{(qt^{n-i};q)_{\lambda_i}}\prod_{1\le i<j\le n}\frac{(qt^{j-i},t^{j-i};q)_{\lambda_i-\lambda_j}}{(t^{j-i+1},qt^{j-i-1};q)_{\lambda_i-\lambda_j}}, \end{aligned} \tag{3.4}$$

for any $n\ge l(\lambda)$. (It is easy to check that the last expression indeed does not depend on n.)

The Macdonald polynomials factor nicely under "principal specialization" [**18**, p. 343, Example 5],

$$P_\lambda(1,t,\dots,t^{N-1};q,t)=t^{\sum_i(i-1)\lambda_i}\prod_{1\le i<j\le N}\frac{(t^{j-i+1};q)_{\lambda_i-\lambda_j}}{(t^{j-i};q)_{\lambda_i-\lambda_j}}. \tag{3.5}$$

A similar formula holds for principally specialized Q_λ, by combining (3.3), (3.4), and (3.5).

We mention two particular useful facts that hold in case of a finite set of variables $X=\{x_1,\dots,x_n\}$ (see [**18**, p. 323, Eq. (4.10), and p. 325, Eq. (4.17)]):

$$P_\lambda(x_1,\dots,x_n;q,t)=0,\qquad\text{if } l(\lambda)>n, \tag{3.6}$$

and

$$P_{(\lambda_1,\lambda_2,\dots,\lambda_n)}(x_1,\dots,x_n;q,t)=(x_1\dots x_n)^{\lambda_n}P_{(\lambda_1-\lambda_n,\lambda_2-\lambda_n,\dots,\lambda_{n-1}-\lambda_n,0)}(x_1,\dots,x_n;q,t). \tag{3.7}$$

There exists [**18**, p. 327] an automorphism $\omega_{q,t}={\omega_{t,q}}^{-1}$ of Sym such that

$$\omega_{q,t}(Q_\lambda(q,t))=P_{\lambda'}(t,q),\qquad \omega_{q,t}(g_k(q,t))=e_k, \tag{3.8}$$

with $\lambda^{'}$ the partition conjugate to λ, whose parts are given by $m_k(\lambda^{'}) = \lambda_k - \lambda_{k+1}$. In particular [**18**, p. 329, Eq. (5.5)], the Macdonald polynomials associated with a row or a column partition are given by

$$P_{1^k}(q,t) = e_k, \qquad Q_{1^k}(q,t) = \frac{(t;t)_k}{(q;t)_k}\, e_k,$$
$$P_{(k)}(q,t) = \frac{(q;q)_k}{(t;q)_k}\, g_k(q,t), \qquad Q_{(k)}(q,t) = g_k(q,t).$$

The parameters q, t being kept fixed, we shall often write P_μ or Q_μ for $P_\mu(q,t)$ or $Q_\mu(q,t)$.

3.1. Pieri formula. Let $u_1, \ldots, u_n$ be n indeterminates and $\mathbb{N}$ the set of nonnegative integers. For $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{N}^n$ let $|\theta| = \sum_{i=1}^n \theta_i$ and define

$$d^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n) = \prod_{k=1}^n \frac{(t, q^{|\theta|+1}u_k;q)_{\theta_k}}{(q, q^{|\theta|}tu_k;q)_{\theta_k}} \prod_{1\le i<j\le n} \frac{(tu_i/u_j, q^{-\theta_j+1}u_i/tu_j;q)_{\theta_i}}{(qu_i/u_j, q^{-\theta_j}u_i/u_j;q)_{\theta_i}}.$$

The Macdonald polynomials satisfy a Pieri formula which generalizes the classical Pieri formula for Schur functions [**18**, p. 73, Eq. (5.16)]. This generalization was obtained by Macdonald [**18**, p. 331], and independently by Koornwinder [**12**].

Most of the time this Pieri formula is stated in combinatorial terms (as a sum over "horizontal strips"). In [**16**, Th. 3.1] it was formulated in "analytic" terms:

Theorem 3.1. *Let $\lambda = (\lambda_1, ..., \lambda_n)$ be an arbitrary partition with length $\le n$ and $\lambda_{n+1} \in \mathbb{N}$. Let $u_i = q^{\lambda_i - \lambda_{n+1}} t^{n-i}$, for $1 \le i \le n$. We have*

$$Q_{(\lambda_1,\ldots,\lambda_n)}\, Q_{(\lambda_{n+1})} = \sum_{\theta\in\mathbb{N}^n} d^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)}.$$

The Pieri formula defines an infinite transition matrix. Indeed, let $\mathsf{Sym}(n+1)$ denote the algebra of symmetric polynomials in $n+1$ independent variables with coefficients in $\mathbb{Q}(q,t)$. Then [**18**, p. 313] the Macdonald polynomials $\{Q_\lambda, \, l(\lambda) \le n+1\}$ form a basis of $\mathsf{Sym}(n+1)$, and so do the products $\{Q_\mu Q_{(m)}, \, l(\mu) \le n, \, m \ge 0\}$.

3.2. A recursion formula. Again, let $u_1, \ldots, u_n$ be n indeterminates. For $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{N}^n$ define

$$\begin{aligned} c^{(q,t)}_{\theta}(u_1,\ldots,u_n) = {} & \prod_{i=1}^n t^{\theta_i} \frac{(q/t, qu_k;q)_{\theta_i}}{(q, qtu_i;q)_{\theta_i}} \prod_{1\le i<j\le n} \frac{(qu_i/tu_j, q^{-\theta_j}tu_i/u_j;q)_{\theta_i}}{(qu_i/u_j, q^{-\theta_j}u_i/u_j;q)_{\theta_i}} \\ & \times \frac{1}{\prod_{1\le i<j\le n}(q^{\theta_i}u_i - q^{\theta_j}u_j)} \det_{1\le i,j\le n}\left[\left(q^{\theta_i}u_i\right)^{n-j} \left(1 - t^{j-1}\frac{1-q^{\theta_i}tu_i}{1-q^{\theta_i}u_i} \prod_{s=1}^n \frac{u_s - q^{\theta_i}u_i}{tu_s - q^{\theta_i}u_i}\right)\right]. \end{aligned}$$

The following recursion formula for Macdonald polynomials was proved in [**16**, Th. 4.1] by inverting the Pieri formula in Theorem 3.1 by utilizing a special case of the multidimensional matrix inverse in [**16**, Th. 2.6].

Theorem 3.2. *Let $\lambda = (\lambda_1, ..., \lambda_{n+1})$ be an arbitrary partition with length $\le n+1$. Let $u_i = q^{\lambda_i - \lambda_{n+1}} t^{n-i}$, for $1 \le i \le n$. We have*

$$Q_{(\lambda_1,\ldots,\lambda_{n+1})} = \sum_{\theta\in\mathbb{N}^n} c^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n)\, Q_{(\lambda_{n+1}-|\theta|)}\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n)}.$$

In the case $n = 1$, i.e. for partitions of length two, Theorem 3.2 reads

$$Q_{(\lambda_1,\lambda_2)} = \sum_{\theta\in\mathbb{N}} c^{(q,t)}_{\theta}(u)\, Q_{(\lambda_2-\theta)}\, Q_{(\lambda_1+\theta)},$$

with $u=q^{\lambda_1-\lambda_2}$ and

$$\begin{aligned} c_\theta^{(q,t)}(u) &= t^\theta \frac{(q/t,qu;q)_\theta}{(q,qtu;q)_\theta}\Big(1-\frac{(1-tq^\theta u)}{(1-q^\theta u)}\frac{(u-q^\theta u)}{(tu-q^\theta u)}\Big) \\ &= t^\theta \frac{(q/t,qu;q)_\theta}{(q,qtu;q)_\theta}\frac{(t-1)}{(t-q^\theta)}\frac{(1-q^{2\theta}u)}{(1-q^\theta u)} \\ &= t^\theta \frac{(1/t,u;q)_\theta}{(q,qtu;q)_\theta}\frac{(1-q^{2\theta}u)}{(1-u)}. \end{aligned}$$

This special case is due to Jing and Józefiak [**9**]. On the other hand, in the case $n=2$, i.e. for partitions of length three, Theorem 3.2 reduces to a formula stated by Lassalle [**13**].

Application of the automorphism $\omega_{q,t}$ to Theorem 3.2, while taking into account (3.8), gives the following equivalent result (cf. [**16**, Th. 4.2]).

THEOREM 3.3. *Let $\lambda=(1^{m_1},2^{m_2},\dots,(n+1)^{m_{n+1}})$ be an arbitrary partition consisting of parts at most equal to $n+1$. Define $u_i=q^{n-i}t^{\sum_{j=i}^n m_j}$, for $1\le i\le n+1$. We have*

$$P_\lambda=\sum_{\theta\in\mathbb{N}^n} c^{(t,q)}_{\theta_1,\dots,\theta_n}(u_1,\dots,u_n)\,e_{m_{n+1}-|\theta|}\,P_{(1^{m_1+\theta_1-\theta_2},\dots,(n-1)^{m_{n-1}+\theta_{n-1}-\theta_n},n^{m_n+m_{n+1}+\theta_n})}.$$

4. Macdonald symmetric functions indexed by partitions with complex parts

We use the recursion in Theorem 3.2 now to *define* Macdonald symmetric functions Q_λ when $\lambda=(\lambda_1,\dots,\lambda_n)$ is *any* sequence of complex numbers. One difficulty is to properly define the one row case. (Another issue is convergence, since we will be considering nonterminating sums.)

Kadell [**10**] used the classical definition of a Schur function in terms of a ratio of alternants to extend Schur functions to partitions with complex parts. (Independently, Danilov and Koshevoy [**4**] define "continuous Schur functions" by a multidimensional integral, with respect to a Lebesgue measure in $\mathbb{R}^{n(n-1)/2}$, over all points of a particular polytope, and show by an inductive argument that these functions generalize the ratio of alternants formula for Schur functions. In fact, Danilov and Koshevoy's continuous Schur functions correspond exactly to Kadell's Schur functions indexed by partitions with *real* parts.) We want to stress that our proposed extension of Macdonald polynomials to complex parts (see below) does not reduce to Kadell's extension of Schur functions when $q=t$.

We shall begin with a finite number of variables, say $X=\{x_0,x_1,\dots,x_r\}$. (For convenience, we start to label X with 0 here). First, consider m to be a nonnegative integer. By appealing to the q-binomial theorem in (2.2) it follows from taking coefficients of u^m in the generating function in (3.2) that the one row Macdonald polynomials $Q_{(m)}(X;q,t)=g_m(X;q,t)$ can be written in the following explicit form:

$$\begin{aligned} g_m &= \sum_{\substack{k_0,\dots,k_r\ge 0\\ k_0+\dots+k_r=m}}\prod_{i=0}^r \frac{(t;q)_{k_i}}{(q;q)_{k_i}}x_i^{k_i} \\ &= \sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le k_1+\dots+k_r\le m}} \frac{(t;q)_{m-(k_1+\dots+k_r)}}{(q;q)_{m-(k_1+\dots+k_r)}}\,x_0^{m-(k_1+\dots+k_r)}\prod_{i=1}^r\frac{(t;q)_{k_i}}{(q;q)_{k_i}}x_i^{k_i}. \end{aligned}$$

Although we do not need it here, we mention that for $r=1$ and $x_0x_1=1$, the g_m reduce to the continuous q-ultraspherical polynomials of degree m in the argument $(x_0+x_1)/2$, considered in [**6**, Ex. 1.29].

We rewrite the above expression yet further, using the short notation $|\mathbf{k}|=k_1+\dots+k_r$, and obtain the following explicit form:

$$g_m=\frac{(t;q)_m}{(q;q)_m}x_0^m\sum_{\substack{k_1,\dots,k_r\ge 0\\ 0\le|\mathbf{k}|\le m}}\frac{(q^{-m};q)_{|\mathbf{k}|}}{(q^{1-m}/t;q)_{|\mathbf{k}|}}\prod_{i=1}^r\frac{(t;q)_{k_i}}{(q;q)_{k_i}}\left(\frac{qx_i}{tx_0}\right)^{k_i}. \tag{4.1}$$

Using the definition

$$(a;q)_c=\frac{(a;q)_\infty}{(aq^c;q)_\infty}$$

(recall $|q| < 1$) for any complex number c, we propose the following definition for a one row complex Macdonald function[1]:

$$Q_{(c)} = g_c = \frac{(tx_0;q)_c}{(q;q)_c}\frac{(q/tx_0;q)_{-c}}{(q/t;q)_{-c}} \sum_{k_1,\dots,k_r\geq 0} \frac{(q^{-c};q)_{|\mathbf{k}|}}{(q^{1-c}/t;q)_{|\mathbf{k}|}} \prod_{i=1}^{r} \frac{(t;q)_{k_i}}{(q;q)_{k_i}} \left(\frac{qx_i}{tx_0}\right)^{k_i}, \tag{4.2}$$

which converges (if the series does not terminate) for $|qx_i/tx_0| < 1$ $(1 \leq i \leq r)$.

We emphasize that (4.2) is *not* an analytic continuation of (4.1). In fact, (4.2) is neither analytic in c, nor in q^c (in any domain around the origin). Another problem is that g_c is not symmetric in all the x_i $(0 \leq i \leq r)$ but only in the last r of the x_i $(1 \leq i \leq r)$. Indeed, already for $r = 1$, the ${}_2\phi_1$ transformation $g_c(x_0, x_1) = g_c(x_1, x_0)$ turns out to be false if c is not an integer.

There are other possibilities to extend g_m to complex numbers. By our definition (4.2), if c is not a nonnegative integer and $q = t$ (the Schur function case), then we get $g_c = \infty$, which is different from Kadell's [**10**, Eq. (2.1)] $s_{(c)}(x_0) = x_0^c = e^{c\ln(x_0)}$. On the other hand, if we would have defined g_c by (4.1) (for complex $m = c$; relaxing the restriction $|\mathbf{k}| \leq m$ of summation), our definition would have also not matched Kadell's in the $q = t$ case since after letting $q = t$ we would be left with a product of geometric series on the right-hand side. However, our particular choice of (4.2) is motivated by some nice properties, among which are (4.8) and (4.10).

Since $g_c(x_0, x_1, \dots, x_r, 0){=}g_c(x_0, x_1, \dots, x_r)$, we may let $r \to \infty$ (compare to [**18**, p. 41]). In the following, we relax the restriction of X being finite. Thus, we allow $r \in \mathbb{N} \cup \infty$.

Having provided a definition of $Q_{(c)}$ for any complex number c, it is now straightforward to extend Theorem 3.2 to Macdonald functions indexed by sequences of complex numbers. Let $\lambda = (\lambda_1, ..., \lambda_{n+1})$ be an arbitrary sequence of complex numbers. We do *not* require $n + 1 \leq |X|$. For any $1 \leq i \leq n + 1$ define $u_i = q^{\lambda_i-\lambda_{n+1}} t^{n-i}$. Then $Q_\lambda(X; q, t)$ is defined recursively by (4.2) and

$$Q_{(\lambda_1,\dots,\lambda_{n+1})} = \sum_{\theta\in\mathbb{N}^n} c^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1, \dots, u_n)\, Q_{(\lambda_{n+1}-|\theta|)}\, Q_{(\lambda_1+\theta_1,\dots,\lambda_n+\theta_n)}, \tag{4.3}$$

where $c^{(q,t)}_{\theta_1,\dots,\theta_n}(u_1, \dots, u_n)$ is the same as in Section 3.2.

While the (finite) recursion in Theorem 3.2 was proved by inverting the known Pieri formula for Macdonald polynomials indexed by partitions, (4.3) now *defines* Macdonald functions in the general case. The expansion in (4.3) is in general infinite and converges (when it does not terminate) for $|q|, |qx_i/tx_0| < 1$ $(i \geq 1)$. As a matter of formal manipulations (using multidimensional inverse relations), the equivalence of Theorems 3.2 and 3.1 (Pieri formula) for these Macdonald functions of complex parts is immediate.

We do not know whether the complex Q_λ form a family of orthogonal functions, nor whether they are eigenfunctions of the Macdonald operator (or some reasonable extension of this operator). These questions, among others, wait for investigation. What at all makes these extended objects interesting, then? In fact, it turns out that for these "complex Macdonald functions" a generalization of (at least) one of the so-called Macdonald (ex-)conjectures holds; in particular, they satisfy an explicit *evaluation formula*.

Observe that we have departed from the algebra of symmetric functions in $X = \{x_0, x_1, \dots\}$. We are working in another algebra. It is necessary to provide some details.

It is convenient to make the following definitions. For a complex number c, introduce the following complex "q, t-powers" of x:

$$x^{[c]} = x^{[c;q,t]} = \frac{(tx;q)_c}{(t;q)_c}\frac{(q/tx;q)_{-c}}{(q/t;q)_{-c}}. \tag{4.4}$$

Note that if k is an integer then $x^{[k]} = x^k$. More generally, $x^{[c+k]} = x^{[c]}x^k$.

Next, using "q, t-powers", extend the definition of power sums to complex numbers c:

$$p_c(x_0, x_1, \dots) = \sum_{i\geq 0} x_i^{[c]}. \tag{4.5}$$

(We will actually only need the one-variable case $p_c(x_0) = x_0^{[c]}$ here.) As usual, this definition may be extended to multiindices, $p_{(c_1,\dots,c_n)} = p_{c_1} \dots p_{c_n}$.

[1]We call these new objects *functions* as they are not anymore polynomials. However, note that the term *Macdonald functions* is also used to denote other objects, namely *modified Bessel functions of the second kind* [**24**]. (The latter have nothing to do with algebraist I. G. Macdonald who introduced the $P_\lambda(X; q, t)$.)

Let $\mathbb{C}$ denote the set of complex numbers. The algebra we are considering is (algebraically) generated by the uncountably infinite set of products

$$\{p_c(x_0)\, p_s(x_1,x_2,\dots)|\, c\in\mathbb{C}, s\in\mathbb{N}-0\}$$

with coefficients in $\mathbb{C}((q,t))$. Note that the above set of products is not an algebraic basis as we do not have algebraic independence (e.g., $(p_{c_1}p_{s_1})(p_{c_2}p_{s_2})=(p_{c_1}p_{s_2})(p_{c_2}p_{s_1})$). We denote this algebra by $\widehat{\mathsf{Sym}}$.

There is a $\widehat{\mathsf{Sym}}$-extension of the homomorphism $\varepsilon_{u,t}$, defined in [**18**, p. 338, Eq. (6.16)], which acts nicely on Q_λ. For an indeterminate u, define a homomorphism $\widehat{\varepsilon}_{u,t}:\widehat{\mathsf{Sym}}\to C((q,t))$ by

$$\widehat{\varepsilon}_{ut,t}\big[p_c(x_0)p_s(x_1,x_2,\dots)\big]=u^{[c]}\,\frac{1-u^s}{1-t^s} \tag{4.6}$$

for each complex number c and positive integer s. In particular, if u is replaced by t^r, we have

$$\widehat{\varepsilon}_{t^{r+1},t}\big[p_c(x_0)p_s(x_1,x_2,\dots)\big]=(t^r)^{[c]}\,\frac{1-t^{rs}}{1-t^s}=p_c(t^r)p_s(t^{r-1},\dots,t,1)$$

and hence for any $f\in\widehat{\mathsf{Sym}}$

$$\widehat{\varepsilon}_{t^{r+1},t}(f)=f(t^r,t^{r-1},\dots,t,1). \tag{4.7}$$

(Compare this to the usual $\varepsilon_{t^{r+1},t}(f)=f(t^r,t^{r-1},\dots,t,1)$, for any $f\in\mathsf{Sym}$.)

We now have the *evaluation formula*

$$\widehat{\varepsilon}_{u,t}Q_{(\lambda_1,\dots,\lambda_n)}=\prod_{i=1}^n\frac{(u;q)_{\lambda_i}}{(qt^{n-i};q)_{\lambda_i}}\frac{(q/u;q)_{-\lambda_i}}{(qt^{i-1}/u;q)_{-\lambda_i}}\prod_{1\le i<j\le n}\frac{(qt^{j-i};q)_{\lambda_i-\lambda_j}}{(qt^{j-i-1};q)_{\lambda_i-\lambda_j}}, \tag{4.8}$$

where $\lambda_i\in\mathbb{C}$. We will present a pure basic hypergeometric proof (which reduces to a new proof of the usual analytic continued principal specialization formula for Q_λ if λ is a partition).

We proceed by induction on n. For $n=1$ we first consider $\widehat{\varepsilon}_{t^{r+1},t}Q_{(c)}$ with $Q_{(c)}$ given in (4.2). By the $a_i\mapsto t$, $u_i\mapsto u^i$, $i=1,\dots n$, $b\mapsto q^{-c}$, and $c\mapsto q^{1-c}/t$ case of Proposition 2.1, we see that

$$\begin{aligned}\widehat{\varepsilon}_{t^{r+1},t}Q_{(c)}&=\frac{(t^{r+1};q)_c}{(q;q)_c}\frac{(q/t^{r+1};q)_{-c}}{(q/t;q)_{-c}}\sum_{k_1,\dots,k_r\ge0}\frac{(q^{-c};q)_{|\mathbf{k}|}}{(q^{1-c}/t;q)_{|\mathbf{k}|}}\prod_{i=1}^r\frac{(t;q)_{k_i}}{(q;q)_{k_i}}\left(qt^{-1-i}\right)^{k_i}\\&=\frac{(t^{r+1};q)_c}{(q;q)_c}\frac{(q/t^{r+1};q)_{-c}}{(q/t;q)_{-c}}\frac{(q^{1-c}/t^{r+1},q/t;q)_\infty}{(q^{1-c}/t,q/t^{r+1};q)_\infty}=\frac{(t^{r+1};q)_c}{(q;q)_c},\end{aligned}$$

which is exactly the $u=t^{r+1}$ case of the right-hand side of (4.8) for $n=1$. Since this holds for $r=0,1,2,\dots$, by analytic continuation we may replace t^{r+1} by u, which establishes the $n=1$ case of (4.8). For the inductive step, we assume (4.8) for partitions λ with $l(\lambda)\le n$. Apply $\widehat{\varepsilon}_{u,t}$ to both sides of the recursion formula (4.3) and use the inductive hypothesis to simplify the summand. We are done if we can show that the sum evaluates to

$$\prod_{i=1}^{n+1}\frac{(u;q)_{\lambda_i}}{(qt^{n+1-i};q)_{\lambda_i}}\frac{(q/u;q)_{-\lambda_i}}{(qt^{i-1}/u;q)_{-\lambda_i}}\prod_{1\le i<j\le n+1}\frac{(qt^{j-i};q)_{\lambda_i-\lambda_j}}{(qt^{j-i-1};q)_{\lambda_i-\lambda_j}}.$$

This, however, follows by an application of the multivariable ${}_6\phi_5$ summation in Corollary 2.2 (after performing the substitutions $a\mapsto t$, $b\mapsto q^{\lambda_{n+1}}ut^{1-n}$, $c_i\mapsto t$, $u_i\mapsto q^{\lambda_i-\lambda_{n+1}}t^{n-i}$, $1\le i\le n$, and $M\mapsto\lambda_{n+1}$). □

There is a well-known duality formula for Macdonald polynomials (cf. [**18**, p. 332, Eq. (6.6)]),

$$\frac{Q_\lambda(q^\mu t^\delta)}{Q_\lambda(t^\delta)}=\frac{Q_\mu(q^\lambda t^\delta)}{Q_\mu(t^\delta)}, \tag{4.9}$$

for partitions λ and μ of length $\le n$, where

$$Q_\lambda(q^\mu t^\delta)=Q_\lambda(q^{\mu_1}t^{n-1},q^{\mu_2}t^{n-2},\dots,q^{\mu_n}).$$

We do not know (at present) whether this relation still holds for arbitrary complex sequences $\lambda_1,\lambda_2,\dots$, and $\mu_1,\mu_2,\dots$. However, it does hold if the length of the partitions is one. Namely, if $\lambda=(c)$, $\mu=(d)$ are one row complex partitions, we have

$$\frac{Q_{(c)}(q^dt,1)}{Q_{(c)}(t,1)}=\frac{(q;q)_c}{(t^2;q)_c}\frac{(tx_0;q)_c}{(q;q)_c}\frac{(q/tx_0;q)_{-c}}{(q/t;q)_{-c}}\sum_{k\ge0}\frac{(q^{-c},t;q)_k}{(q^{1-c}/t,q;q)_k}\left(\frac{q^{1-d}}{t^2}\right)^k$$

$$= \frac{(tx_0;q)_c}{(t^2;q)_c} \frac{(q/tx_0;q)_{-c}}{(q/t;q)_{-c}} \; {}_2\phi_1\left[\begin{matrix} q^{-c}, t \\ q^{1-c}/t \end{matrix}; q, q^{1-d}/t^2\right].$$

The duality is now an immediate consequence of the iterate of Heine's transformation [**6**, Eq. (III.2)],

$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(c/b, bz;q)_\infty}{(c,z;q)_\infty} \; {}_2\phi_1\left[\begin{matrix} abz/c, b \\ bz \end{matrix}; q, \frac{c}{b}\right],$$

valid for $\max(|z|, |c/b|) < 1$.

To prevent possible misconception, we note that the well-known property valid for Macdonald polynomials indexed by partitions, $Q_\lambda(x_1, \dots, x_r) = 0$ if $l(\lambda) > r$ (see (3.3) and (3.6)), does *not* hold in the general complex case. For instance, if $X = \{x\}$ contains only one variable, then

$$Q_{(\lambda_1,\dots,\lambda_n)}(x) = \prod_{i=1}^n \frac{(tx;q)_{\lambda_i}}{(qt^{n-i};q)_{\lambda_i}} \frac{(q/tx;q)_{-\lambda_i}}{(qt^{i-2};q)_{-\lambda_i}} \prod_{1\le i<j\le n} \frac{(qt^{j-i};q)_{\lambda_i-\lambda_j}}{(qt^{j-i-1};q)_{\lambda_i-\lambda_j}}, \tag{4.10}$$

where $\lambda_i \in \mathbb{C}$. This formula (which can be proved by induction, similar to the above proof of (4.8)) is indeed independent from the representation of λ, i.e. we may choose $(\lambda_1, \dots, \lambda_n) = (\lambda_1, \dots, \lambda_n, 0, \dots, 0)$, adding an arbitrary number of zeros at the end of sequence. It is clear from (4.10) that if λ is a usual integer partition, then

$$Q_{(\lambda_1,\dots,\lambda_n)}(x) = \frac{(t;q)_{\lambda_1}}{(q;q)_{\lambda_1}} x^{\lambda_1} \delta_{\lambda_2 0} \dots \delta_{\lambda_n 0}$$

(where we were using (4.4) and $x^{[k]} = x^k$ for integer k).

In this section, we extended the Macdonald polynomials Q_λ to arbitrary sequences $\lambda = (\lambda_1, \dots, \lambda_n)$ of complex numbers. To give such an extension for P_λ one may simply invoke $Q_\lambda = b_\lambda P_\lambda$, see (3.3), with the known explicit expression of $b_\lambda = b_\lambda(q,t)$, extended to complex sequences λ. (This does not mean that we necessarily assume $b_\lambda = \; < Q_\lambda, Q_\lambda >_{q,t}$ beforehand. Nevertheless, the latter equality should conjecturally still hold, for some suitable $\widehat{\mathsf{Sym}}$-extension of the inner product $<,>_{q,t}$.) To utilize Theorem 3.3 to define P_λ in the complex case does not make sense since the indexing partitions are given there in the form $\lambda = (1^{m_1}, 2^{m_2}, \dots, (n+1)^{m_{n+1}})$, and the multiplicities m_i have no meaning for partitions with complex parts. (Already for compositions λ one would run into trouble here.)

Again, at the present moment we do not know whether the Macdonald functions for partitions with complex parts form a family of orthogonal functions. They very well may be orthogonal, possibly with respect to a $\widehat{\mathsf{Sym}}$ variant of the inner product in [**18**, p. 372, Eq. (9.10)]. In view of (4.8) and (4.10), some nice properties do exist, which provides some evidence that these new objects merit further investigation.

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