

Combinatorial realisation of Hall-Littlewood polynomials at $t = -1$

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ABSTRACT. A generalisation is given of Hall-Littlewood polynomials, $P_\lambda(\mathbf{x}; t)$, to the case of an arbitrary simple Lie algebra \mathfrak{g} . For $t = 0$ and $t = 1$ these polynomials are just the appropriate Weyl group symmetric monomial function and Weyl's character of an irreducible representation, respectively. Here the case $t = -1$ is discussed in some detail. First a factorisation result is established for all \mathfrak{g} , and then a combinatorial realisation of $P_\lambda(\mathbf{x}; -1)$ is offered for both $gl(n)$ and $sp(2n)$ in terms of certain Q -functions that are shown to be Weyl group symmetric. The Q -functions are defined in terms of primed shifted tableaux. In addition a lattice path interpretation is provided in the form of determinantal expansions.

RÉSUMÉ. Nous donnons un généralisation des polynômes Hall-Littlewood, $P_\lambda(\mathbf{x}; t)$, à une algèbre Lie simple arbitraire \mathfrak{g} . À $t = 0$ et $t = 1$ ces polynômes ne sont que la fonction symétrique monomiale du groupe Weyl et le caractère Weyl d'une représentation irréductible. Nous discutons ici en détail le cas $t = -1$. Nous établissons d'abord un résultat de factorisation pour tout choix de \mathfrak{g} , et ensuite nous dérivons une réalisation combinatoire de $P_\lambda(\mathbf{x}; -1)$ pour $gl(n)$ et $sp(2n)$ en termes de certaines Q -fonctions qui sont symétriques dans le groupe Weyl. Les Q -fonctions sont définies en termes des tableaux primés et décalés. De plus, nous donnons une interprétation de chemins de treillis sous forme d'expansion de déterminant.

1. Introduction

Let \mathfrak{g} be a simple Lie algebra of rank r , with Cartan subalgebra \mathfrak{h} and dual \mathfrak{h}^* . Let the corresponding Weyl group be W and let the sets of positive roots and simple roots be denoted by Δ^+ and Π , respectively. For each $w \in W$, let $\ell(w)$ be the minimal length of w when expressed as a word in the generators, w_i with $i = 1, 2, \dots, r$, of the Weyl group W .

An arbitrary weight vector, $\mu \in \mathfrak{h}^*$, is said to be integral, dominant integral or strongly dominant integral if $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}$, $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}_{> 0}$ for all $\alpha \in \Pi$. In such cases we write $\mu \in P$, P^+ or P^{++} , respectively. The Weyl vector $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ is such that $\langle \rho, \alpha \rangle = 1$ for all $\alpha \in \Pi$. It follows that for each $\mu \in P^{++}$ we can write $\mu = \lambda + \rho$ with $\lambda \in P^+$.

Let $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ be a sequence of orthonormal vectors spanning \mathfrak{h}^* and let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ with $x_k = e^{\epsilon_k}$ for $k = 1, 2, \dots, n$. Then for each $\mu \in P^+$ we have $e^\mu = \mathbf{x}^\mu = (x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n})$. In this situation the action of the Weyl group is defined by $w(e^\mu) = e^{w(\mu)} = \mathbf{x}^{w(\mu)}$. The corresponding monomial Weyl-symmetric function is defined by:

$$(1.1) \quad m_\mu(\mathbf{x}) = \sum_{w \in W^\mu = W/W_\mu} \mathbf{x}^{w(\mu)},$$

where $W_\mu = \{w \in W \mid w(\mu) = \mu\}$, and W^μ is a set of minimal length left coset representatives of W with respect to W_μ .

Macdonald [6, 7] introduced a two-parameter family of orthogonal polynomials, $P_\mu(\mathbf{x}; q, t)$ with $\mu \in P^+$, uniquely characterised by the conditions:

$$(1.2) \quad \begin{aligned} \text{(i)} \quad & P_\mu(\mathbf{x}; q, t) = m_\mu(\mathbf{x}) + \sum_{\nu < \mu} a_{\nu\mu}(q, t) m_\nu(\mathbf{x}) \quad \text{with} \quad a_{\nu\mu} \in \mathbb{Q}(q, t) \\ \text{(ii)} \quad & \langle P_\mu(\mathbf{x}; q, t), P_\nu(\mathbf{x}; q, t) \rangle = 0 \quad \text{if} \quad \nu \neq \mu, \end{aligned}$$

2000 *Mathematics Subject Classification*. Primary 17B10; Secondary 05E10.

Key words and phrases. algebraic combinatorics, symmetric functions, ECO systems, enumeration, partitions.

for a suitably defined partial order relation $<$ and symmetric scalar product $\langle \cdot, \cdot \rangle$ [6, 7].

Here we confine our attention to the case $q \rightarrow 0$, for which the Macdonald polynomials $P_\mu(\mathbf{x}; 0, t)$ coincide with the Hall-Littlewood polynomials $P_\mu(\mathbf{x}; t)$. For these cases there exists the explicit formula [5, 6, 7]:

$$(1.3) \quad P_\mu(\mathbf{x}; t) = \frac{1}{W_\mu(t)} \sum_{w \in W} w \left(\mathbf{x}^\mu \prod_{\alpha \in \Delta^+} \left(\frac{1 - t \mathbf{x}^{-\alpha}}{1 - \mathbf{x}^{-\alpha}} \right) \right),$$

with

$$(1.4) \quad W_\mu(t) = \sum_{w \in W_\mu} t^{\ell(w)}.$$

The special case $t = 1$ gives the monomial Weyl-symmetric functions

$$(1.5) \quad P_\mu(\mathbf{x}; 1) = \frac{1}{|W_\mu|} \sum_{w \in W} w(\mathbf{x}^\mu) = \sum_{w \in W^\mu = W/W_\mu} \mathbf{x}^{w(\mu)} = m_\mu(\mathbf{x}).$$

For $t = 0$ we obtain

$$(1.6) \quad P_\mu(\mathbf{x}; 0) = \sum_{w \in W} w \left(\mathbf{x}^\mu \prod_{\alpha \in \Delta^+} \left(\frac{1}{1 - \mathbf{x}^{-\alpha}} \right) \right) = \sum_{w \in W} w \left(\frac{\mathbf{x}^{\mu+\rho}}{a_\rho(\mathbf{x})} \right),$$

where $a_\rho(\mathbf{x})$ is Weyl's denominator function

$$(1.7) \quad a_\rho(\mathbf{x}) = \mathbf{x}^\rho \prod_{\alpha \in \Delta^+} (1 - \mathbf{x}^{-\alpha}) = \prod_{\alpha \in \Delta^+} (\mathbf{x}^{\alpha/2} - \mathbf{x}^{-\alpha/2}).$$

However, $w(a_\rho(\mathbf{x})) = (-1)^{\ell(w)} a_\rho(\mathbf{x})$, so that

$$(1.8) \quad P_\mu(\mathbf{x}; 0) = \frac{1}{a_\rho(\mathbf{x})} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{w(\mu+\rho)} = \frac{a_{\mu+\rho}(\mathbf{x})}{a_\rho(\mathbf{x})}.$$

This serves to define $P_\mu(\mathbf{x}; 0)$ for all weights μ , but if μ is an integral dominant weight, that is $\mu \in P^+$, then this is nothing other than Weyl's formula for the character of the irreducible representation V^μ of \mathfrak{g} having highest weight μ . That is to say, for each $\mu \in P^+$, we have

$$(1.9) \quad \text{ch } V^\mu(\mathbf{x}) = P_\mu(\mathbf{x}; 0).$$

The special case $\mu = 0$, for which $\text{ch } V^0 = 1$, gives Weyl's denominator identity:

$$(1.10) \quad a_\rho(\mathbf{x}) = \mathbf{x}^\rho \prod_{\alpha \in \Delta^+} (1 - \mathbf{x}^{-\alpha}) = \sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{w(\rho)}.$$

It follows from the second form of $a_\rho(\mathbf{x})$ given here that $a_{2\rho}(\mathbf{x}) = a_\rho(\mathbf{x}^2)$, where \mathbf{x}^2 is obtained from \mathbf{x} by squaring every component. This implies in turn that

$$(1.11) \quad \text{ch } V^\rho(\mathbf{x}) = P_\rho(\mathbf{x}; 0) = \frac{a_{2\rho}(\mathbf{x})}{a_\rho(\mathbf{x})} = \frac{a_\rho(\mathbf{x}^2)}{a_\rho(\mathbf{x})} = \frac{\mathbf{x}^{2\rho}}{\mathbf{x}^\rho} \prod_{\alpha \in \Delta^+} \left(\frac{1 - \mathbf{x}^{-2\alpha}}{1 - \mathbf{x}^{-\alpha}} \right) = \mathbf{x}^\rho \prod_{\alpha \in \Delta^+} (1 + \mathbf{x}^{-\alpha}).$$

The final expression can be thought of as a rather simple deformation of Weyl's denominator function (1.7).

The interesting case $t = -1$ of $P_\mu(\mathbf{x}; t)$ defies, in general, such a simple analysis, if for no other reason than the singular nature of $1/W_\mu(-1)$ preventing in some cases the direct use of the formula (1.3). However, if μ is strongly integral dominant, that is $\mu \in P^{++}$ then $w(\mu) = \mu$ if and only if $w = 1$, the identity element of W . In such a case $W_\mu(-1) = 1$. This allows us to see from (1.3) that for all $\mu \in P^{++}$

$$(1.12) \quad \begin{aligned} P_\mu(\mathbf{x}; -1) &= \sum_{w \in W} w \left(\mathbf{x}^\mu \prod_{\alpha \in \Delta^+} \left(\frac{1 + \mathbf{x}^{-\alpha}}{1 - \mathbf{x}^{-\alpha}} \right) \right) = \sum_{w \in W} w \left(\mathbf{x}^\mu \frac{\mathbf{x}^{2\rho}}{(\mathbf{x}^\rho)^2} \prod_{\alpha \in \Delta^+} \left(\frac{(1 - \mathbf{x}^{-2\alpha})}{(1 - \mathbf{x}^{-\alpha})^2} \right) \right) \\ &= \frac{a_\rho(\mathbf{x}^2)}{(a_\rho(\mathbf{x}))^2} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{w(\mu)} = \frac{1}{a_\rho(\mathbf{x})} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{w(2\rho)} \frac{1}{a_\rho(\mathbf{x})} \sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{w(\mu)}, \end{aligned}$$

where use has been made of (1.10) with \mathbf{x} replaced by \mathbf{x}^2 . If we now set $\mu = \lambda + \rho$, with $\lambda \in P^+$ by virtue of our assumption that $\mu \in P^{++}$, it then follows from (1.8) and (1.9) that

$$(1.13) \quad P_{\lambda+\rho}(\mathbf{x}; -1) = P_\rho(\mathbf{x}; 0) P_\lambda(\mathbf{x}; 0) = \text{ch } V^\rho(\mathbf{x}) \text{ch } V^\lambda(\mathbf{x}).$$

where $\mathcal{PD}^\rho(n)$ is the subset of $\mathcal{QD}^\rho(n)$ consisting only of those primed triangular tableaux PD having no primed entries on the main diagonal.

However, with $\rho = (n, n-1, \dots, 1)$ and positive roots $\alpha = \epsilon_i - \epsilon_j$ for $1 \leq i < j \leq n$, it follows from (1.11) and the above that

$$(2.10) \quad s_\rho(\mathbf{x}) = \mathbf{x}^\rho \prod_{1 \leq i < j \leq n} (1 - x_i^{-1} x_j) = \prod_{1 \leq i \leq n} x_i \prod_{1 \leq i < j \leq n} (x_i + x_j) = \sum_{PD \in \mathcal{PD}^\rho(n)} \mathbf{x}^{\text{wgt}(PD)}.$$

As mentioned in the Introduction, there exists a bijection [3] between $PST \in \mathcal{PST}^{\lambda+\rho}(n)$ and pairs (PD, T) with $PD \in \mathcal{PD}(n)$ and $T \in \mathcal{T}^\lambda(n)$. This is illustrated by the map:

$$(2.11) \quad PST = \begin{array}{cccccccc} 1 & 1 & 1 & 2' & 2 & 2 & 3 & 3 & 5 \\ & 2 & 2 & 3' & 3 & 4' & 5' & 5 & 6' \\ & & 3 & 3 & 4' & 4 & 5' & 6 \\ & & & 4 & 5' & 5 & 5 \\ & & & & 5 & 6' & 6 \\ & & & & & & 6 \end{array} \longrightarrow (PD, T) = \begin{array}{ccccccc} 1 & 2' & 1 & 4' & 5' & 6' \\ & 2 & 3' & 2 & 5' & 2 \\ & & 3 & 4' & 3 & 3 \\ & & & 4 & 5' & 6' \\ & & & & 5 & 5 \\ & & & & & 6 \end{array}, \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 5 & 5 \\ 4 & 6 \\ 5 \\ 6 \end{array}$$

To effect this map one moves all primed entries k' north west into their own column, that is the k th column, taking k' in turn equal to $1', 2', \dots, n'$ and dealing with each k' in turn from top to bottom. This is done by a sequence of jeu de taquin moves in the form of transpositions of k' with their north or west unprimed neighbours, i or j , respectively:

$$(2.12) \quad \begin{array}{c} \boxed{j} \\ \boxed{i \quad k'} \end{array} \longrightarrow \begin{array}{c} \boxed{k'} \\ \boxed{i \quad j} \end{array} \quad \text{if } i \leq j;$$

$$\begin{array}{c} \boxed{j} \\ \boxed{i \quad k'} \end{array} \longrightarrow \begin{array}{c} \boxed{j} \\ \boxed{k' \quad i} \end{array} \quad \text{if } i > j.$$

The chosen move is the unique one that ensures that the unprimed entries remain weakly increasing across rows and strongly increasing down columns. Of course k' must move west if it is already in the 1st row but not yet in the k th column. When it reaches the k th column it may still move north by means of a transposition of the type:

$$(2.13) \quad \begin{array}{c} \boxed{i} \\ \boxed{k'} \end{array} \longrightarrow \begin{array}{c} \boxed{k'} \\ \boxed{i} \end{array}$$

unless i is in the i th row, in which case the k' moves no further. It has been shown [3] that each k' always reaches a position off the main diagonal in the k th column and that in the first n columns all unprimed entries lie in their own row. Moreover, by reversing the individual jeu de taquin steps it has been shown that this map is bijective. Finally, it should be noted that the map is weight preserving since all the steps involve interchanging the positions of entries, without any alteration of their value.

By virtue of the combinatorial realisation of $P_{\lambda+\rho}(\mathbf{x})$ with $\ell(\lambda + \rho) = n$ that is implied by (2.6), and those of $s_\lambda(\mathbf{x})$ and $s_\rho(\mathbf{x})$ given in (2.2) and (2.10), respectively, it follows from the existence of the above bijection that for all partitions λ :

$$(2.14) \quad P_{\lambda+\rho}(\mathbf{x}; -1) = P_{\lambda+\rho}(\mathbf{x}) = s_\rho(\mathbf{x}) s_\lambda(\mathbf{x}).$$

3. The $sp(2n)$ case

In the case $\mathfrak{g} = sp(2n)$, which has rank n , we work in the n -dimensional space spanned by Euclidean basis vectors ϵ_i with $i = 1, 2, \dots, n$. The set of positive roots is given by $\Delta^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\epsilon_i \mid 1 \leq i \leq n\}$ and $\rho = n\epsilon_1 + (n-1)\epsilon_2 + \dots + \epsilon_n = (n, n-1, \dots, 1)$. We set $x_k = e^{\epsilon_k}$ and $\bar{x}_k = x_k^{-1} = e^{-\epsilon_k}$ for $k = 1, 2, \dots, n$.

Each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with no more than n non-vanishing parts specifies a finite-dimensional irreducible representation V^λ of $sp(2n)$. Its character is given by $\text{ch } V^\lambda(\mathbf{x}) = sp_\lambda(\mathbf{x})$, where $sp_\lambda(\mathbf{x})$ may be referred to as a symplectic Schur function. This may be given a combinatorial definition as follows [4]. Each symplectic tableau $T \in \mathcal{T}^\lambda(n, \bar{n})$ is filling of the boxes of F^λ with entries from the ordered set $\{\bar{1} < 1 < \bar{2} < 2 < \dots < \bar{n} < n\}$ weakly increasing across rows and strictly increasing down columns, with no entry k or \bar{k}

It is far from obvious that the function $P_\lambda^{sp}(\mathbf{x})$ does, as claimed, coincides with the $t = -1$ specialisation of the $sp(2n)$ Hall-Littlewood polynomial $P_\lambda(\mathbf{x}; t)$, or even that it is symmetric with respect to the Weyl group of $sp(2n)$, which includes both all the permutations of the components of \mathbf{x} and any combination of the inversions $x_k \mapsto x_k^{-1}$ with $k \in \{1, 2, \dots, n\}$. However, in the case that λ is a partition with n non-vanishing distinct parts these properties can be established, thanks to the existence of the factorisation property (1.13).

To see how this comes about it is necessary to introduce, as our final set of tableaux, the set $\mathcal{QD}^\rho(n, \bar{n})$. Each $PD \in \mathcal{QD}^\rho(n, \bar{n})$ is a tableau of shifted shape SF^ρ where $\rho = (n, n-1, \dots, 1)$, with entries taken from the set $\{\bar{1}', \bar{1}, 1', 1, \bar{2}', \bar{2}, 2', 2, \dots, \bar{n}', \bar{n}, n', n\}$, in which each entry k or \bar{k} lies in the k th row and each entry k' or \bar{k}' lies in the k th column. The weight of such a primed tableau is given by $\text{wgt}(PD) = \kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)$, where κ_k equals the total number of entries k or k' minus the total number of entries \bar{k} or \bar{k}' . Typically for $n = 5$ we have

$$(3.8) \quad PD = \begin{array}{ccccc} \bar{1} & 1 & \bar{3}' & 4' & \bar{1} \\ & 2 & 2 & 2 & \bar{2} \\ & & \bar{3}' & 3 & \bar{3} \\ & & & 4' & 4 \\ & & & & \bar{5}' \end{array} \quad \text{wgt}(PD) = (-1, 2, -2, -1, -1).$$

It follows from our definition of $\mathcal{QD}^\rho(n, \bar{n})$ that

$$(3.9) \quad \sum_{PD \in \mathcal{QD}^\rho(n, \bar{n})} \mathbf{x}^{\text{wgt}(PD)} = \prod_{1 \leq i < j \leq n} (x_i + \bar{x}_i + x_j + \bar{x}_j).$$

As usual this expression is divisible by $2^{\ell(\rho)} = 2^n$ and we have

$$(3.10) \quad \sum_{PD \in \mathcal{PD}^\rho(n, \bar{n})} \mathbf{x}^{\text{wgt}(PD)} = \prod_{1 \leq i \leq n} (x_i + \bar{x}_i) \prod_{1 \leq i < j \leq n} (x_i + \bar{x}_i + x_j + \bar{x}_j),$$

where $\mathcal{PD}^\rho(n, \bar{n})$ is the subset of $\mathcal{QD}^\rho(n, \bar{n})$ consisting of those PD that have no primes on their main diagonal.

However, with $\rho = (n, n-1, \dots, 1)$ and positive roots $\epsilon_i \pm \epsilon_j$ for $1 \leq i < j \leq n$ and $2\epsilon_i$ for $1 \leq i \leq n$, it follows from (1.11) that

$$(3.11) \quad \begin{aligned} sp_\rho(\mathbf{x}) &= \mathbf{x}^\rho \prod_{1 \leq i \leq n} (1 + x_i^{-2}) \prod_{1 \leq i < j \leq n} (1 + x_i^{-1}x_j)(1 + x_i^{-1}x_j^{-1}) \\ &= \prod_{1 \leq i \leq n} (x_i + x_i^{-1}) \prod_{1 \leq i < j \leq n} (x_i + x_i^{-1} + x_j + x_j^{-1}) = \sum_{PD \in \mathcal{PD}^\rho(n, \bar{n})} \mathbf{x}^{\text{wgt}(PD)}, \end{aligned}$$

where in the last step use has been made of the fact that $\bar{x}_k = x_k^{-1}$ for all $k = 1, 2, \dots, n$.

This time the required bijection [3] is one between $PST \in \mathcal{QST}^{\lambda+\rho}(n, \bar{n})$ and pairs (PD, T) with $PD \in \mathcal{QD}^\rho(n, \bar{n})$ and $T \in \mathcal{T}^\lambda(n, \bar{n})$. This is illustrated in the case $n = 5$ by the map:

$$(3.12) \quad QST = \begin{array}{cccccc} \bar{1} & 1' & \bar{2}' & \bar{3}' & 3 & \bar{4} & \bar{4} & 5 \\ & \bar{2}' & \bar{2} & \bar{3} & \bar{3}' & 4' & 4 & 4 \\ & & \bar{3} & 3 & \bar{4} & 5' & 5 & 5 \\ & & & 4 & 4 & & & \\ & & & & 5' & & & \end{array} \quad \longrightarrow \quad (PD, T) = \begin{array}{cccc} 1' & 1 & \bar{3}' & 4' & \bar{1} \\ & \bar{2}' & \bar{2} & 4' & 5' \\ & & \bar{3} & \bar{4}' & \bar{3} \\ & & & 4 & 4 \\ & & & & 5' \end{array}, \quad \begin{array}{ccc} \bar{1} & \bar{4} & \bar{4} & 5 \\ 3 & 4 & 4 & \\ \bar{4} & 5 & 5 & \end{array}$$

The map is effected by moving all primed entries k' and \bar{k}' into their own column by means of a sequence of *jeu de taquin* moves, dealing first with all \bar{k}' 's and the all k' 's. The allowed moves include all those described in the $gl(n)$. These are applied first to all \bar{k}' 's, and in doing so no obstacles are encountered. However, in moving the k' 's one may encounter a final move in which the destination site in the i th row and k th column is already occupied by a \bar{k}' . In this case we use the following transposition:

$$(3.13) \quad \begin{array}{|c|c|} \hline \bar{k}' & k' \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|} \hline i & \bar{i} \\ \hline \end{array}$$

Similarly, to avoid a pair i and \bar{i} in the k th column, which would be non-standard in that they cannot both be in the i th row, one replaces such a pair by a \bar{k}' and k' pair as below:

$$(3.14) \quad \begin{array}{|c|} \hline \bar{i} \\ \hline i \\ \hline \end{array} \longrightarrow \begin{array}{|c|} \hline k' \\ \hline \bar{k}' \\ \hline \end{array}$$

It is this last transformation that forces one to allow primed entries on the main diagonal, and thus establish the bijection at the level of $\mathcal{QST}^{\lambda+\rho}(n, \bar{n})$, rather than $\mathcal{PST}^{\lambda+\rho}(n, \bar{n})$. Having done this it has been shown that each primed entry k' or \bar{k}' reaches the k th column, and that in the first n columns each unprimed entry lies in its own row. Just like the previous moves, the two new moves described above are weight preserving since the contribution to the weight of any pair of entries $k\bar{k}$ or $k'\bar{k}'$ is zero. It follows that the map is a weight preserving bijection. This allows us to see that

$$(3.15) \quad \sum_{QST \in \mathcal{QST}^{\lambda+\rho}(n, \bar{n})} \mathbf{x}^{\text{wgt}(QST)} = \sum_{PD \in \mathcal{PD}^{\rho}(n, \bar{n})} \mathbf{x}^{\text{wgt}(PD)} \sum_{T \in \mathcal{T}^{\lambda}(n, \bar{n})} \mathbf{x}^{\text{wgt}(T)}.$$

Taking into account the fact that $\ell(\lambda + \rho) = \ell(\rho) = n$ this leads to the factorisation:

$$(3.16) \quad P_{\lambda+\rho}^{sp}(\mathbf{x}) = sp_{\rho}(\mathbf{x}) sp_{\lambda}(\mathbf{x}).$$

Thanks to the general factorisation identity (1.13), this suffices to prove our Conjecture 3.1 in any case for which λ has length $\ell(\lambda) = n$.

4. Symmetry properties

4.1. The Weyl symmetry of Schur Q -functions. Before embarking on a direct proof of the fact that $Q_{\lambda}(\mathbf{x})$, as defined combinatorially by (2.4), is a symmetric function of the components of \mathbf{x} , it is worth both extending the definition to the case of skew Q -functions $Q_{\lambda/\mu}(\mathbf{x})$ and introducing supersymmetric skew Schur functions $s_{\tau/\sigma}(\mathbf{x}/\mathbf{y})$.

Dealing first with the latter, if τ and σ are partitions such that $F^{\tau/\sigma} = F^{\tau} \setminus F^{\sigma}$ is a skew Young diagram, then for any $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_n)$, we have

$$(4.1) \quad s_{\tau/\sigma}(\mathbf{x}/\mathbf{y}) = \sum_{T \in \mathcal{T}^{\tau/\sigma}(m/n)} \prod_{i=1}^m x_i^{\#i \in T} \prod_{j=1}^n y_j^{\#j' \in T},$$

where $\mathcal{T}^{\tau/\sigma}(m/n)$ is the set of all skew supertableaux, T , of shape $F^{\tau/\sigma}$ with entries taken from the set $\{1, \dots, m, 1', \dots, n'\}$, subject to the ordering $1 < 2 < \dots < m < 1' < 2' < \dots < n'$, with unprimed entries increasing weakly across rows and strictly down columns, and primed entries increasing strictly across rows and weakly down columns. This function is supersymmetric in the sense that it is symmetric under any permutation of (x_1, \dots, x_m) , symmetric under any permutation of (y_1, \dots, y_n) and is independent of z if we set $x_i = z$ and $y_j = -z$ for any $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. It is a remarkable, and very useful fact that $s_{\tau/\sigma}(\mathbf{x}/\mathbf{y})$ is independent of the order relation on its entries [5], see Ex23, p90. In particular, in the case $m = n$ it is convenient to adopt the ordering $1' < 1 < 2' < 2 < \dots < n' < n$.

Similarly, if λ and μ are partitions whose parts are distinct, such that $SF^{\lambda/\mu} = SF^{\lambda} \setminus SF^{\mu}$ is a skew shifted Young diagram, then for any $\mathbf{x} = (x_1, x_2, \dots, x_n)$ we have

$$(4.2) \quad Q_{\lambda/\mu}(\mathbf{x}) = \sum_{PST \in \mathcal{QST}^{\lambda/\mu}(n)} \prod_{k=1}^n x_k^{(\#k \in PST) + (\#k' \in PST)},$$

where $\mathcal{QST}^{\lambda/\mu}(n)$ is the set of all primed semistandard shifted skew tableaux, PST , of shape $SF^{\lambda/\mu}$, with entries taken from the set $\{1', 1, 2', 2, \dots, n', n\}$ subject to the ordering $1' < 1 < 2' < 2 < \dots < n' < n$, weakly increasing across rows and down columns, with no two k' 's in the same row, and no two k 's in the same column.

It should then be noted that each $PST \in \mathcal{QST}^{\lambda}(n)$, with λ a partition whose parts are all distinct, has a structure consisting of a sequence of ribbon strips $\text{str}_k(PST)$ with $k = 1, 2, \dots, n$. This structure is specified by means of a sequence $\text{Seq}(\lambda)$ of the form $\lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \dots \subseteq \lambda^{(n)}$, with $\lambda^{(0)} = 0$ and $\lambda^{(n)} = \lambda$,

This shows, once again, that $Q_{\lambda^{(b)}/\lambda^{(a-1)}}(x_a, x_b)$ is symmetric under the interchange of x_a and x_b .

It follows that all contributions $Q_{\lambda^{(b)}/\lambda^{(a-1)}}(x_a, x_b)$ to $Q_\lambda(\mathbf{x})$ are symmetric under the interchange of $x_a = x_k$ and $x_b = x_{k+1}$. Applying this in the cases $k = 1, 2, \dots, n-1$ is sufficient to prove our required result that $Q_\lambda(\mathbf{x})$, as defined by (2.4), is symmetric under all permutations of the components of \mathbf{x} .

4.2. The Weyl symmetry of symplectic Q -functions. Now, the aim is to show that $Q_\lambda^{sp}(\mathbf{x})$, as defined combinatorially by (3.5), is not only a symmetric function of the components of \mathbf{x} but also symmetric with respect to any combination of the inversions $x_k \mapsto x_k^{-1}$ with $k \in \{1, 2, \dots, n\}$. Once again, it is convenient to extend the definition to the case of skew symplectic Q -functions as follows. If λ and μ are partitions whose parts are distinct, such that $SF^{\lambda/\mu} = SF^\lambda \setminus SF^\mu$ is a skew shifted Young diagram, then for any $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ we let

$$(4.9) \quad Q_{\lambda/\mu}(\mathbf{x}, \mathbf{y}) = \sum_{QST \in \mathcal{QST}^{\lambda/\mu}(n, \bar{n})} \prod_{x_k}^{n} (\#k \in QST) + (\#k' \in QST) \prod_{y_k}^{n} (\#\bar{k} \in QST) + (\#\bar{k}' \in QST),$$

where $\mathcal{QST}^{\lambda/\mu}(n, \bar{n})$ is the set of all primed semistandard shifted symplectic skew tableaux, QST , of shape $SF^{\lambda/\mu}$, with entries taken from the set $\{\bar{1}', \bar{1}, 1', 1, \bar{2}', \bar{2}, 2', 2, \dots, \bar{n}', \bar{n}, n', n\}$ subject to the ordering $\bar{1}' < \bar{1} < 1' < 1 < \bar{2}' < \bar{2} < 2' < 2 < \dots < \bar{n}' < \bar{n} < n' < n$, weakly increasing across rows and down columns, with no two k 's or \bar{k} 's in the same row, and no two ks or $\bar{k}s$ in the same column. This allows us to define

$$(4.10) \quad Q_{\lambda/\mu}^{sp}(\mathbf{x}) = Q_{\lambda/\mu}(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{QST \in \mathcal{QST}^{\lambda/\mu}(n, \bar{n})} x_k^{\text{wgt}(QST)},$$

where $\text{wgt}(QST) = (\#k \in QST) + (\#k' \in QST) - (\#\bar{k} \in QST) - (\#\bar{k}' \in QST)$.

It should then be noted that each $QST \in \mathcal{QST}^{\lambda}(n, \bar{n})$, with λ a partition whose parts are all distinct, has a structure consisting of a sequence of ribbon strips $\text{str}_{\bar{k}}(QST)$ and $\text{str}_k(QST)$ with $k = 1, 2, \dots, n$. This structure is specified by means of a sequence $\text{Seq}(\lambda)$ of the form $\lambda^{(0)} \subseteq \lambda^{(\bar{1})} \subseteq \lambda^{(1)} \subseteq \dots \subseteq \lambda^{(\bar{n})} \subseteq \lambda^{(n)}$, with $\lambda^{(0)} = 0$ and $\lambda^{(n)} = \lambda$, where each $\lambda^{(a)}$ is a partition all of whose parts are distinct, and $\lambda^{(a)}/\lambda^{(a-1)}$ is a skew partition in the shape of the ribbon strip $\text{str}_a(PST)$ for $a = \bar{1}, 1, \bar{2}, 2, \dots, \bar{n}, n$. Here $a-1$ is to be interpreted as the symbol immediately to the left of a in the sequence $(0, \bar{1}, 1, \bar{2}, 2, \dots, \bar{n}, n)$, that is to say $a-1 = k-1$ if $a = \bar{k}$ and $a-1 = \bar{k}$ if $a = k$. With this notation,

$$(4.11) \quad Q_\lambda^{sp}(\mathbf{x}) = \sum_{\text{Seq}(\lambda)} \prod_{a=\bar{1}}^n Q_{\lambda^{(a)}/\lambda^{(a-1)}}(x_a).$$

Moreover, with similar definitions of $a+1$ and $a+2$, one and two steps, respectively, to the right of a in the sequence $(0, \bar{1}, 1, \bar{2}, 2, \dots, \bar{n}, n)$, we have

$$(4.12) \quad Q_\lambda^{sp}(\mathbf{x}) = \sum_{\text{Seq}(\lambda)} Q_{\lambda^{(a-1)}}(x_1, \dots, x_{a-1}) Q_{\lambda^{(a+1)}/\lambda^{(a-1)}}(x_a, x_{a+1}) Q_{\lambda^{(n)}/\lambda^{(a+1)}}(x_{a+2}, \dots, x_n).$$

It follows that the contribution to $Q_\lambda(\mathbf{x})$ arising from the entries $a', a, (a+1)', (a+1)$ in all those $QST \in \mathcal{QST}^\lambda(n, \bar{n})$ whose ribbon strip structure is specified by $\text{Seq}(\lambda)$ is given by $Q_{\lambda^{(a+1)}/\lambda^{(a-1)}}(x_a, x_{a+1})$.

In the case $a = \bar{k}$, so that $b = a+1 = k$ and $a-1 = k-1$, then at most one of $\{a', a, b', b\}$ may appear on the main diagonal. It follows that $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ is always a skew Young diagram $F^{\tau/\sigma}$ for some pair of partitions τ and σ . This is illustrated in the case of our example (3.12) by:

$$(4.13) \quad \begin{array}{cccccc} & & & & \bar{4} & \bar{4} \\ & & & & & \\ & & & & 4' & 4 & 4 \\ & & & & & & \\ & & & & \bar{4} & & \\ & & & & & & \\ & & & & 4 & 4 & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{array} \longrightarrow \begin{array}{cccc} & & & \bar{4} & \bar{4} \\ & & & & \\ & & & 4' & 4 & 4 \\ & & & & \\ & & & \bar{4} & \\ & & & & \\ & & & 4 & 4 \end{array}$$

where it can be seen that $\tau = (5, 5, 2, 2)$ and $\sigma = (3, 2, 1)$. It then follows that in each such case

$$(4.14) \quad Q_{\lambda^{(k)}/\lambda^{(k-1)}}(x_{\bar{k}}, x_k) = s_{\tau/\sigma}(x_{\bar{k}}, x_k/x_{\bar{k}}, x_k).$$

Since $s_{\tau/\sigma}(x_a, x_b/y_a, y_b)$ is symmetric under the interchange of x_a and x_b , and of y_a and y_b , it follows that $Q_{\lambda^{(k)}/\lambda^{(k-1)}}(x_{\bar{k}}, x_k)$ is symmetric under the interchange of $x_{\bar{k}}$ and x_k . This result is valid for all $k = 1, 2, \dots, n$

Thanks to the identification $x_{\bar{k}} = x_k^{-1}$ for $k = 1, 2, \dots, n$, this implies that $Q_{\lambda}^{sp}(\mathbf{x})$ is symmetric with respect to any combination of the inversions $x_k \mapsto x_k^{-1}$ with $k \in \{1, 2, \dots, n\}$.

On other hand, if $a = k$, so that $b = a + 1 = \overline{k+1}$ and $a - 1 = \bar{k}$, the situation is the same as that of $a = k$ and $b = k + 1$ for $gl(n)$. Two types of case occur, namely those for which $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ is a skew Young diagram $F^{\tau/\sigma}$ for a pair of partitions τ and σ , and those for which $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ contains a connected sequence of pairs of boxes on d consecutive diagonals, starting with the main diagonal, linked to at least one box, the initial box, of a skew Young diagram $F^{\tau/\sigma}$ for a pair of partitions τ and σ . The analysis goes through exactly as in the $gl(n)$ case, giving either

$$(4.15) \quad Q_{\lambda^{(\overline{k+1})}/\lambda^{\bar{k}}}(x_k, x_{\overline{k+1}}) = s_{\tau/\sigma}(x_k, x_{\overline{k+1}}/x_k, x_{\overline{k+1}})$$

or

$$(4.16) \quad Q_{\lambda^{(\overline{k+1})}/\lambda^{\bar{k}}}(x_k, x_{\overline{k+1}}) = 2(x_k, x_{\bar{k}})^d s_{\tau/\sigma}(x_k, x_{\overline{k+1}}/x_k, x_{\overline{k+1}}).$$

In both cases $Q_{\lambda^{(\overline{k+1})}/\lambda^{\bar{k}}}(x_k, x_{\overline{k+1}})$ is symmetric under the interchange of x_k and $x_{\overline{k+1}}$. This result is valid for all $k = 1, 2, \dots, n - 1$.

Now, returning to the symplectic Schur Q -function $Q_{\lambda}^{sp}(\mathbf{x})$ itself, its dependence on x_k and x_{k+1} can be isolated by means of the following decomposition

$$(4.17) \quad Q_{\lambda}^{sp}(\mathbf{x}) = \sum_{\text{Seq}(\lambda)} Q_{\lambda^{(k-1)}}(x_{\bar{1}}, \dots, x_{k-1}) Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_{\bar{k}}, x_k, x_{\overline{k+1}}, x_{k+1}) Q_{\lambda^{(n)}/\lambda^{(k+1)}}(x_{\overline{k+2}}, \dots, x_n),$$

where

$$(4.18) \quad \begin{aligned} Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_{\bar{k}}, x_k, x_{\overline{k+1}}, x_{k+1}) &= Q_{\lambda^{(k)}/\lambda^{(k-1)}}(x_{\bar{k}}, x_k) Q_{\lambda^{\overline{k+1}}/\lambda^{(k)}}(x_{\overline{k+1}}) Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_{k+1}) \\ &= Q_{\lambda^{(\bar{k})}/\lambda^{(k-1)}}(x_{\bar{k}}) Q_{\lambda^{(k+1)}/\lambda^{(\bar{k})}}(x_k, x_{\overline{k+1}}) Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_{k+1}) \\ &= Q_{\lambda^{(\bar{k})}/\lambda^{(k-1)}}(x_{\bar{k}}) Q_{\lambda^{(k)}/\lambda^{(\bar{k})}}(x_k) Q_{\lambda^{(k+1)}/\lambda^{(k)}}(x_{\overline{k+1}}, x_{k+1}). \end{aligned}$$

The fact that the first, second and third expressions are symmetric with respect to the transpositions $x_{\bar{k}} \leftrightarrow x_k$, $x_k \leftrightarrow x_{\overline{k+1}}$ and $x_{\overline{k+1}} \leftrightarrow x_{k+1}$, respectively, ensures that $Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_{\bar{k}}, x_k, x_{\overline{k+1}}, x_{k+1})$ is symmetric with respect to all permutations of its arguments, including the permutation $(x_{\bar{k}}, x_k, x_{\overline{k+1}}, x_{k+1}) \mapsto (x_{\overline{k+1}}, x_{k+1}, x_{\bar{k}}, x_k)$. This implies in turn that $Q_{\lambda}^{sp}(\mathbf{x})$ is invariant under the transposition $x_k \leftrightarrow x_{k+1}$. Since this is true for all $k = 1, 2, \dots, n - 1$, it follows that $Q_{\lambda}^{sp}(\mathbf{x})$ is symmetric with respect to all permutations of the components of \mathbf{x} . Combining this with the invariance of $Q_{\lambda}^{sp}(\mathbf{x})$ under inversions, it follows that $Q_{\lambda}^{sp}(\mathbf{x})$ is Weyl group invariant, as required.

5. Lattice path approach

It is well-known that tableaux translate nicely to lattice paths. In the case of Schur Q -function, Stembridge [12] has provided a lattice path interpretation and used this to derive a pfaffian result for $Q_{\lambda}(\mathbf{x})$. Hamel [1] extended this approach to obtain pfaffians for more general decompositions of tableaux. Hamel also included a determinantal expression for $Q_{\lambda}(\mathbf{x})$ due to Okada [8] and showed an extension to it too. Here we derive a determinantal expression for $Q_{\lambda}^{sp}(\mathbf{x})$ and prove it using lattice paths.

The lattice path grid is defined as follows. Label the y -axis with $0, \frac{1}{2}, \overline{1}, \bar{1}, \overline{1\frac{1}{2}}, 1', 1, 1\frac{1}{2}, \overline{2}, \bar{2}, \overline{2\frac{1}{2}}, 2', 2, \dots$. Define lattice paths with three types of permissible steps: up-vertical steps that increase the y -coordinate by 1; horizontal steps at unprimed levels that increase the x -coordinate by 1; and up-diagonal steps from unprimed levels to primed levels that increase the x -coordinate by 1 and increase the y -coordinate by 1.

For each tableau with m parts we can construct an m -tuple of nonintersecting lattice paths. There will be one path for each row in the tableau. Construct a path as follows: if a box in the row contains an i or \bar{i} and is at coordinates (a, b) in the tableau, put a horizontal step from $(a - b, i)$ to $(a - b + 1, i)$ (resp. $(a - b, \bar{i})$ to $(a - b + 1, \bar{i})$); if a box in the row contains an i' or \bar{i}' and is at coordinates (a, b) in the tableau, put an up-diagonal step from $(a - b, \bar{i})$ to $(a - b + 1, i')$ (resp. $(a - b, i - 1)$ to $(a - b + 1, \bar{i}')$). If, however, the box is on the main diagonal of the tableau then there is a minor modification. In that case a box containing i' or \bar{i}' causes an up-diagonal step from $(0, i\frac{1}{2})$ to $(1, i')$ (resp. $(0, (i - 1)\frac{1}{2})$ to $(1, \bar{i}')$). These steps are then connected with vertical steps.

Okada [8] proved a plane partition version of the following result using lattice paths. Hamel [1] reinterpreted it in the form described above. In the theorem, a_1, a_2, \dots, a_l is the sequence of elements on the main diagonal of the tableau, and $|a_j| = k$ if $a_j = k$ or k' .

THEOREM 5.1. *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a shifted shape partition. Then*

$$Q_\lambda(\mathbf{x}) = \sum_{a_1 < a_2 < \dots < a_l} \det [x_{|a_j|} Q_{\lambda_i-1}(x_{|a_j|}, x_{|a_j|+1}, \dots)].$$

Here we generalize this theorem to the symplectic case $Q_\lambda^{sp}(\mathbf{x})$. We now take $|a_j| = k$ if $a_j = k, k', \bar{k}$ or $\overline{k'}$.

THEOREM 5.2. *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a shifted shape partition. Then*

$$Q_\lambda^{sp}(\mathbf{x}) = \sum_{a_1 < a_2 < \dots < a_l} \left(\delta_{a_j, \bar{k}} \det [x_{|a_j|}^{-1} Q_{\lambda_i-1}(x_{|a_j|}^{-1}, x_{|a_j|}, x_{|a_j|+1}, \dots)] \right. \\ \left. + \delta_{a_j, \mathbf{k}} \det [x_{|a_j|} Q_{\lambda_i-1}(x_{|a_j|}, x_{|a_j|+1}^{-1}, x_{|a_j|+1}, \dots)] \right),$$

where $\delta_{a_j, \bar{k}}$ is 1 if a_j is barred and 0 otherwise, and similarly $\delta_{a_j, \mathbf{k}}$ is 1 if a_j is unbarred and 0 otherwise.

The Q generating function in the determinant reflects the fact that with the first element removed from the row, the row is now of size $\lambda_i - 1$ and the second element must have a weight of $|a_j|$ or greater.

Proof: Let a_1, a_2, \dots, a_l be the set of elements in the boxes on the main diagonal of the shifted tableau. Using the lattice path set-up as described above with the additional constraint that the first step in a path must be a step to $(1, a_j)$.

A Lindström-Gessel-Viennot argument will supply the proof. In particular, Theorem 1.2 of Stembridge [12] applies. Summing over all permissible sequences a gives the full generality of the result. This completes the proof.

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