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Demazure atoms

Sarah Mason

Abstract. Demazure characters can be decomposed into smaller pieces, called standard bases by Lascoux and Schützenberger. We prove that these polynomials, which we call Demazure atoms, are the same polynomials obtained from a certain specialization of nonsymmetric Macdonald polynomials. This combinatorial interpretation for Demazure atoms accelerates the computation of the key associated to a given semi-standard Young tableau.

Résumé. Des caractères de Demazure peuvent être décomposés en plus petits morceaux, appelés les bases standard par Lascoux et le Schutzenberger. Nous montrons que ces polynômes, que nous appelons des atomes de Demazure, sont les mêmes polynômes obtenus à partir d'une certaine spécialisation des polynômes nonsymmetric de Macdonald. Cette interprétation combinatoire pour des atomes de Demazure accélère le calcul de la clef associée à un SSYT.

1. Introduction

The Demazure character formula generalizes the Weyl character formula to extremal weight modules. Let $u_{w\lambda}$ be the extremal weight vector of weight $w\lambda$. This vector generates a $U(\mathfrak{n})$-module $U(\mathfrak{n})u_{w\lambda}$. The formal character of this submodule is given by Demazure's character formula [**1**], [**4**].

Key polynomials [**8**] are equivalent to the polynomials obtained from the Demazure character formula. They can be described combinatorially as the sums of the weights of semi-standard Young tableaux whose right key is bounded by a certain key $K(\omega, \lambda)$.

Lascoux and Schützenberger [**6**] study the smallest non-intersecting pieces $\mathfrak{U}(\omega, \lambda)$ of Demazure characters. These polynomials can be defined combinatorially as the sums of the weights of semi-standard Young tableaux whose right key is equal to the key $K(\omega, \lambda)$. Since each semi-standard Young tableau appears in precisely one polynomial $\mathfrak{U}(\omega, \lambda)$, these polynomials form a decomposition of the Schur functions. It is natural to compare this to other Schur function decompositions.

One decomposition of the Schur functions is given by the polynomials $E_\gamma(x; 0, 0)$, which are obtained by setting $q = t = 0$ in the combinatorial formula for nonsymmetric Macdonald polynomials. They are given by weights of *semi-skyline augmented fillings*, which are fillings of composition diagrams with positive integers in such a way that the columns are weakly decreasing and the rows satisfy an inversion condition. Semi-skyline augmented fillings are in bijection with semi-standard Young tableaux and satisfy a variation of the Robinson-Schensted-Knuth algorithm [**7**].

Theorem 1.1. *The Demazure atom $\mathfrak{U}(\omega, \lambda)$ is equal to the specialized Macdonald polynomial $E_{\omega(\lambda)}(X; 0, 0)$.*

We obtain a quick and simple method for computing the right key of a semi-standard Young tableau as a corollary to Theorem 1.1. Begin with a semi-standard Young tableau T and map T to the semi-skyline

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augmented filling $\Psi(T)$ whose weight is equal to that of T. If the shape of $\Psi(T)$ is given by the composition γ, then the right key of T is the key whose weight is given by γ.

2. Demazure characters

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let Φ be the corresponding root system. If $\mathfrak{n}$ is the subalgebra of $\mathfrak{g}$ with basis $X_\alpha \quad (\alpha \in \Phi^+)$, then $U(\mathfrak{n})$ is the universal enveloping algebra. Let $V(\lambda)$ be the irreducible highest-weight module of weight λ. Given an element, ω, of the Weyl group, let $u_{\omega\lambda}$ be the extremal vector of weight $\omega\lambda$ of $V(\lambda)$. Then the formal character of $U(\mathfrak{n})u_{\omega\lambda}$ is given by the Demazure character formula. In this section we describe the explicit formula, using Demazure operators, in the case where $\mathfrak{g}$ is the general linear Lie algebra.

2.1. The Demazure operator. Let P be the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$ and let S_∞ be the permutation group of the positive integers. This group acts on P by permuting the indices of the variables. If s_i be the elementary transposition $(i, i+1)$, define the linear operators ∂_i and π_i as in [**8**] by

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}, \qquad \pi_i = \partial_i x_i. \tag{2.1}$$

Given $\omega \in S_\infty$, one can write ω as a product of elementary transpositions $s_{i_1} s_{i_2} \ldots s_k$. When the number k of transpositions in such a product is minimized, the word $i_1 i_2 \ldots i_k$ is called a *reduced word* for ω. Then $\pi_\omega = \pi_{i_1} \pi_{i_2} \ldots \pi_{i_k}$.

The operator π_ω is precisely the Demazure operator [**1**], [**4**] for the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. One obtains the *Demazure character* corresponding to a partition λ and a permutation ω by applying the operator π_ω to the dominant monomial x^λ.

2.2. An equivalent definition. A *key* is a semi-standard Young tableau such that the set of entries in the $(j+1)^{th}$ column are a subset of the set of entries in the j^{th} column, for all j. There is a bijection [**8**] between compositions and keys given by $\gamma \mapsto key(\gamma)$, where $key(\gamma)$ is the key whose first γ_j columns contain the letter j. To invert this map, send the key T to the composition denoting the content of T.

EXAMPLE 2.1. Let $\gamma = (2, 1, 1, 4, 0, 3)$. Then

$$key(\gamma) = \begin{array}{|c|c|c|c|} \hline 6 \\ \cline{1-1} 4 \\ \cline{1-2} 3 & 6 \\ \cline{1-3} 2 & 4 & 6 \\ \hline 1 & 1 & 4 & 4 \\ \hline \end{array}$$

Let w be any word such that applying the RSK algorithm to w yields the pair of tableaux (P, Q). Then w is *Knuth equivalent* [**5**] to col(P), denoted $w \sim P$. (Here col(P) is the column word obtaining by reading the columns of P from top to bottom, left to right.) There is a unique word v in each equivalence class such that $v = col(T)$ for some semi-standard Young tableau T. Let λ' be the conjugate shape of a partition λ, obtained by reflecting the Ferrers diagram of λ across the line $x = y$. Let w be an arbitrary word such that $w \sim T$ for T of shape λ. Then $colform(w)$ is the composition consisting of the lengths of the subwords in $col(w)$. The word w is said to be *column-frank* if $colform(w)$ is a rearrangement of the nonzero parts of λ'.

EXAMPLE 2.2. (a) Let $w = 3\ 5\ 4\ 2\ 2\ 1$. Then $colform(w) = (1, 3, 2)$. But $w \sim T$, where

$$T = \begin{array}{|c|c|} \hline 5 \\ \cline{1-1} 3 \\ \cline{1-2} 2 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$$

Therefore $\lambda' = (4, 2)$, so w is not column-frank since $(1, 3, 2)$ is not a rearrangement of $(4, 2)$.

(b) Let $v = 4\ 2\ 5\ 3\ 2\ 1$. Then $colform(v) = (2, 4)$. In this case, $v \sim P$, where

$$P = \begin{array}{|c|c|} \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 5 \\ \hline 1 & 2 \\ \hline \end{array}$$

Again $\lambda' = (4, 2)$, so in this case v is column-frank.

Let T be a semi-standard Young tableau of shape λ. The *right key* of T, denoted $K_+(T)$, is the key of shape λ whose j^{th} column is given by the last column of any column-frank word v such that $v \sim T$ and *colform*(v) is of the form $(\ldots, \lambda'_j)$ [**8**]. (For $T = 5\ 3\ 2\ 1 \cdot 4\ 2$ above, we have $K_+(T) = 5\ 4\ 2\ 1 \cdot 4\ 2$.)

Given an arbitrary partition λ and permutation ω, there exists an associated key $K(\omega, \lambda)$ defined as follows. Take the subword of ω consisting of the first λ_1 letters and reorder the letters in decreasing order. This is the first column of $K(\omega, \lambda)$. Then take the first λ_2 letters of ω and order these in decreasing order to get the second column of $K(\omega, \lambda)$. Continuing this way, one derives the word *col*$(K(\omega, \lambda))$ [**6**]. For example, $\omega = 241635$ and $\lambda = (4, 2, 2, 1)$ give the key $6\ 4\ 2\ 1 \cdot 4\ 2 \cdot 4\ 2 \cdot 2$.

The key polynomial $\kappa_{\omega(\lambda)}$ is defined [**8**] as the sum of the weights of all SSYT having right key less than or equal to $K(\omega, \lambda)$. This polynomial is precisely the Demazure character $\pi_\omega(x^\lambda)$ [**6**], so we will use these terms interchangeably.

2.3. Intersections of Demazure characters. Notice that for a fixed partition λ, the semi-standard Young tableaux appearing as weights in the polynomials $\pi_\omega(x^\lambda)$ intersect nontrivially. For example,

$$\pi_{(1,2,3)}(x^{(2,1)}) = \pi_1\pi_2(x_1^2x_2) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_2x_3,$$

$$\pi_{(1)(2,3)}(x^{(2,1)}) = \pi_2(x_1^2x_2) = x_1^2x_2 + x_1^2x_3,$$

$$\pi_{(1,2)(3)}(x^{(2,1)}) = \pi_1(x_1^2x_2) = x_1^2x_2 + x_1x_2^2.$$

In particular, if $\omega = s_i\sigma$, then the SSYTs appearing in $\pi_\sigma(x^\lambda)$ are a subset of those appearing in $\pi_\omega(x^\lambda) = \pi_i\pi_\sigma(x^\lambda)$. Therefore it makes sense to consider the intersections of Demazure characters and their complements.

Consider all of the possible permutation σ such that $\omega = s_i\sigma$ for some i. We study the subset of $\pi_\omega(x^\lambda)$ which does not appear in $\pi_\sigma(x^\lambda)$ for any such σ. These polynomials are precisely the polynomials obtained by replacing the operator π_i by the operator $\overline{\pi_i} = \pi_i - 1$. The operator $\overline{\pi}_i = \overline{\pi}_{s_i}$ is therefore defined by

$$f \longrightarrow (s_i(f) - f)/(1 - x_i/x_{i+1}) = \overline{\pi}_i(f),$$

and $\overline{\pi}_\omega(f) = \overline{\pi}_{i_1}(f)\overline{\pi}_{i_2}(f)\ldots\overline{\pi}_{i_k}(f)$ given any reduced word $s_{i_1}s_{i_2}\ldots s_{i_k}$ for ω. For example, if $f = x_1^2x_2x_3$, then $\overline{\pi}_1 f = \frac{(x_1x_2^2x_3 - x_1^2x_2x_3)}{(1-x_1/x_2)} = x_1x_2^2x_3$.

Lascoux and Schützenberger [**6**] call these polynomials the *standard bases* and prove that the standard basis $\mathfrak{U}(\omega, \lambda)$ equals the sum of the weights of all SSYT having right key equal to $K(\omega, \lambda)$. We retain the notation $\mathfrak{U}(\omega, \lambda)$ but call the polynomials *Demazure atoms* to avoid confusion with different objects referred to as standard bases.

The operators $\overline{\pi}_i$ satisfy the Coxeter relations $\overline{\pi}_i\overline{\pi}_{i+1}\overline{\pi}_i = \overline{\pi}_{i+1}\overline{\pi}_i\overline{\pi}_{i+1}$ and $\overline{\pi}_i\overline{\pi}_j = \overline{\pi}_j\overline{\pi}_i$ for $\|j-i\| > 1$ [**6**]. We can lift the operator $\overline{\pi}_i$ to an operator θ_i on the free algebra by the following process. Given i and a word w in the alphabet $X = (x_1, x_2, \ldots)$, let m be the number of times the letter x_{i+1} occurs in w and let $m+k$ be the number of times the letter x_i occurs in w. Then if $k \geq 0$, w and w^{σ_i} differ by the exchange of a subword x_i^k into x_{i+1}^k. The case where $k < 0$ is not needed in this paper. When $k \geq 0$, define $w\theta_i$ to be the sum of all words in which the subword x_i^k has been changed respectively into $x_i^{k-1}x_{i+1}$, $x_i^{k-2}x_{i+1}^2$, $\ldots$, x_{i+1}^k.

Every partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ has a corresponding *dominant monomial*,

$$x^\lambda = (x_{\lambda_1}\ldots x_2x_1)(x_{\lambda_2}\ldots x_2x_1)\ldots = \prod_i x_i^{\lambda_i},$$

which equals the weight of the *super tableau.* (The super tableau is the SSYT which contains only the entry i in the i^{th} row.)

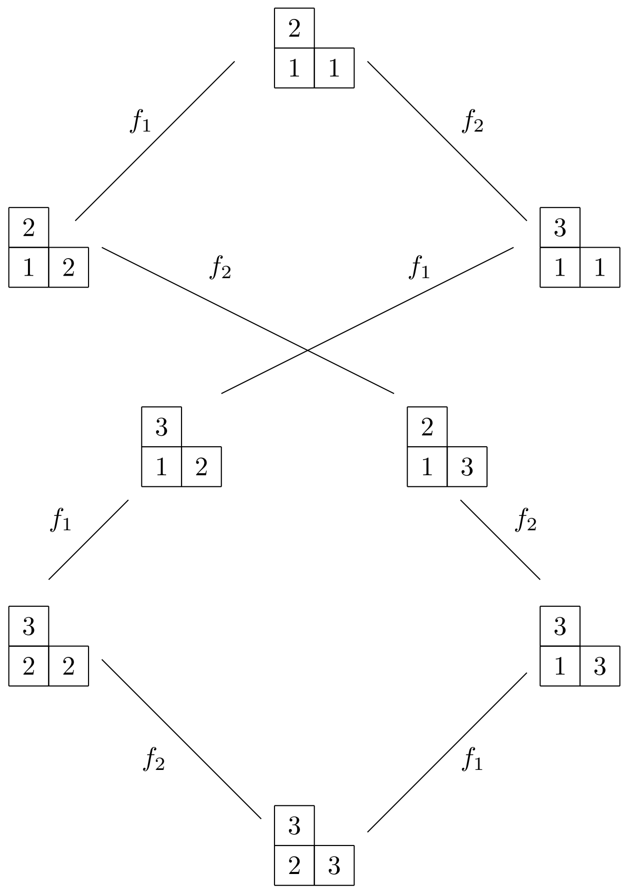


FIGURE 2.1. The crystal graph for $\lambda = (2,1)$.

THEOREM 2.3. *(Lascoux-Schützenberger* [**6**]*) Let x^λ be the dominant monomial corresponding to λ and $s_i s_j \dots s_k$ be any reduced decomposition of a permutation π. Then $\mathfrak{U}(\omega,\lambda) = \theta_i\theta_j...\theta_k x^\lambda$.*

Theorem 2.3 provides an inductive method for constructing the standard basis $\mathfrak{U}(\omega,\lambda)$. Begin with $\mathfrak{U}(id,\lambda)$ and apply θ_i to determine $\mathfrak{U}(\sigma_i,\lambda)$. Then apply θ_j to $\mathfrak{U}(\sigma_i,\lambda)$ to determine $\mathfrak{U}(\sigma_i\sigma_j,\lambda)$. Continue this process until the desired standard basis is obtained.

Lascoux and Schützenberger further break down this procedure to describe a crystal graph structure. We describe the operator f_i needed for this procedure. Let col(T) be the column word corresponding to the semi-standard young tableau T. Change all occurrences of i in col(T) to right parentheses and all occurrences of $i+1$ in col(T) to left parentheses. Ignore all other entries in col(T) and match the parnetheses in the usual manner. If there are no unmatched right parentheses, then $f_i(\text{col}(T)) = \text{col}(T)$. Otherwise replace the rightmost unmatched right parenthesis by a left parenthesis and convert the parentheses back to occurrences of i and $i+1$. The resulting word is $f_i(\text{col}(T))$. Figure 2.3 depicts the crystal graph corresponding to the partition $(2,1)$.

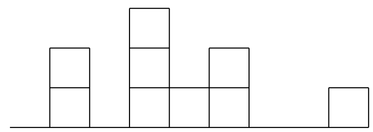
The Demazure character corresponding to $\omega = s_{i_1}s_{i_2}\dots s_{i_k}$ is obtained from this picture by applying the appropriate f_i operators. To see this, begin with the highest SSYT, which corresponds to the monomial x^λ. Apply f_{i_k} until the semi-standard Young tableau stabilizes. The monomials obtained together with the initial monomial is the Demazure character $\kappa_{s_{i_k}(\lambda)}$. Next apply $f_{i_{k-1}}$ to the monomials in $\kappa_{s_{i_k}(\lambda)}$ until the tableaux stabilize to obtain $\kappa_{s_{i_{k-1}}s_{i_k}(\lambda)}$. Continue this procedure to obtain $\kappa_{\omega(\lambda)}$

3. Combinatorial description of the nonsymmetric Schur functions

The nonsymmetric Schur functions are obtained from the nonsymmetric Macdonald polynomials by letting q and t approach infinity. Haglund, Haiman, and Loehr provide a combinatorial formula for nonsymmetric Macdonald polynomials [**3**] which can be specialized to obtain a combinatorial formula for nonsymmetric Schur functions. Several definitions are needed to describe this formula.

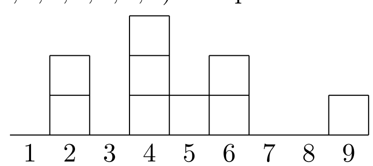
3.1. Semi-skyline augmented fillings. Let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a weak composition of n. The *column diagram of* γ is a figure $dg'(\gamma)$ consisting of n *cells* arranged into columns, as in [**3**]. The i^{th} column contains γ_i cells, and the number of cells in a column is called the *height* of that column. We place a horizontal line across the bottom to denote where the columns begin and end. A cell a in an augmented diagram is denoted $a = (i, j)$, where i is the x-coordinate and j is the y-coordinate.

For example, the figure below is the column diagram of $\gamma = (0, 2, 0, 3, 1, 2, 0, 0, 1)$.

$dg'(\gamma) =$ 

The *augmented diagram* of γ, defined by $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(i, 0) : 1 \leq i \leq m\}$ (where m is the number of parts of γ) is the column diagram with m extra cells adjoined in row 0. In this paper the adjoined row, called the *basement*, always contains the numbers 1 through m in strictly increasing order. Therefore we replace the basement cells with the numbers 1 through m.

The augmented diagram for $\gamma = (0, 2, 0, 3, 1, 2, 0, 0, 1)$ is depicted below.

$\widehat{dg}(\gamma) =$ 

An *augmented filling,* σ, of an augmented diagram $\widehat{dg}(\gamma)$ is a function $\sigma : \widehat{dg}(\gamma) \to \mathbb{Z}_+$, which we picture as an assignment of positive integer entries to the cells of γ. Let $\sigma(i)$ denote the entry in the i^{th} cell of the augmented diagram encountered when $\widehat{dg}(\gamma)$ is read across rows from left to right, beginning at the highest row and working downwards. This ordering of the cells is called the *reading order.* (A cell $a = (i, j)$ is greater than a cell $b = (i', j')$ in the reading order if either $j' > j$ or $j' = j$ and $i' < i$.) The reading word *read*(σ) is obtained by recording the entries in reading order. The *content* of a filling σ is the multiset of entries which appear in the filling.

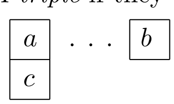
A filling is said to be *non-attacking* if $\sigma(a) = \sigma(b)$ implies that a and b are not in the same row and if a and b are in adjacent rows, the entry in the lower row is weakly to the right of the entry in the higher row.

To describe the nonsymmetric Macdonald polynomials, Haglund, Haiman and Loehr introduce the statistics *Des*(σ) and *Inv*(σ). As in [**?**], a *descent* of σ is a pair of entries $\sigma(a) > \sigma(b)$, where the cell a is directly above b. Thus if $b = (i, j)$, then $a = (i, j + 1)$. Define $Des(\sigma) = \{a \in dg'(\gamma) : \sigma(a) > \sigma(b) \text{ is a descent}\}$. Let

$$\text{maj}(\widehat{\sigma}) = \sum_{a \in \text{Des}(\widehat{\sigma})} (l(a) + 1),$$

where $l(a)$ is the number of cells above a in the column containing a of $dg'(\gamma)$.

Three cells $\{a, b, c\} \in \lambda$ are called a *type A triple* if they are situated as follows



where a and c are in the same row, possibly the first row, possibly with cells between them, b is directly below a, and the height of the column containing a and b is greater than or equal to the height of the column containing c.

Define for $x, y \in \mathbb{Z}_+$

$$I(x, y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases} .$$

Let σ be an augmented filling and let $\{\alpha, \beta, \delta\}$ be the entries of σ in the cells $\{a, b, c\}$, respectively, of a type A triple. The triple $\{a, b, c\}$ is called a *type A inversion triple* if and only if $I(\alpha, \beta) + I(\beta, \delta) - I(\alpha, \delta) = 1$.

Similarly, three cells $\{a, b, c\} \in \lambda$ are a *type B triple* if they are situated as shown,

$$\begin{array}{ccc} & & \boxed{c} \\ \boxed{a} & \cdots & \boxed{b} \end{array}$$

where a and c are in the same row (possibly the basement), possibly with cells between them, b is directly on top of c, and the column containing b and c is strictly taller than the column containing a.

Let σ be an augmented filling and let $\{\alpha, \beta, \delta\}$ be the entries of σ in the cells $\{a, b, c\}$, respectively, of a type B triple. The triple $\{a, b, c\}$ is called a *type B inversion triple* if and only if $I(\delta, \alpha)+I(\alpha, \beta)-I(\delta, \beta) = 1$.

Let $\mathrm{inv}(\widehat{\sigma})$ be the number of type A inversion triples plus the number of type B inversion triples. Let $\mathrm{coinv}(\widehat{\sigma})$ be the number of triples which are not an inversion triple of either type.

THEOREM 3.1. [**3**] *The non-symmetric Macdonald polynomials E_μ are given by the formula*

$$E_\gamma(X; q, t) = \sum_{\substack{\sigma\colon \gamma\to[n] \\ \textit{non-attacking}}} x^\sigma q^{\mathrm{maj}(\widehat{\sigma})} t^{\mathrm{coinv}(\widehat{\sigma})} \prod_{\substack{u\in \mathrm{dg}'(\gamma) \\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u))}} \frac{1-t}{1-q^{l(u)+1}t^{a(u)+1}}, \tag{3.1}$$

where $x^\sigma = \prod_{u\in\mathrm{dg}'(\mu)} x_{\sigma(u)}$.

We are concerned only with the polynomial $E_\gamma(X; 0, 0)$, so setting q and t equal to zero produces the polynomial

$$\begin{aligned} E_\gamma(X;0,0) &= \sum_{\substack{\sigma\colon \gamma\to[n] \\ \text{non-attacking}}} x^\sigma 0^{\mathrm{maj}(\widehat{\sigma})} 0^{\mathrm{coinv}(\widehat{\sigma})} \prod_{\substack{u\in \mathrm{dg}'(\gamma) \\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u))}} \frac{1-0}{1-0^{l(u)+1}0^{a(u)+1}} \\ &= \sum_{\substack{\sigma\colon \gamma\to[n] \\ \text{non-attacking} \\ \mathrm{maj}(\widehat{\sigma})=\mathrm{coinv}(\widehat{\sigma})=0}} x^\sigma \end{aligned}$$

A non-attacking filling F satisfying $\mathrm{maj}(F) = \mathrm{coinv}(F) = 0$ is call a *semi-skyline augmented filling, SSAF*. We provide a simpler definition of a semi-skyline augmented filling.

3.2. Non-attacking fillings. Let F be a semi-skyline augmented filling. The condition $\mathrm{maj}(F) = 0$ implies that

$$\mathrm{maj}(F) = \sum_{a\in\mathrm{Des}(F)} (l(a) + 1) = 0,$$

so this means that $\mathrm{Des}(F) = \emptyset$. The condition $\mathrm{coinv}(F) = 0$ implies that every triple of cells in F must be an inversion triple. Therefore a semi-skyline augmented filling is a non-attacking filling with no descents such that every triple is an inversion triple.

The following two Lemmas demonstrate that any filling satisfying the descent and inversion conditions must be a non-attacking filling.

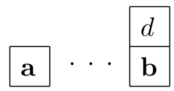
LEMMA 3.2. *Let F be a descentless augmented filling such that every triple of F is an inversion triple. If the cells a and b are in the same row of F, then $F(a) \neq F(b)$.*

PROOF. Suppose a and b are in the same row of F. We may assume that a is to the left of b. Assume first that the column containing a is weakly taller than the column containing b. The cell a is directly on top of some cell c, so $\{a, b, c\}$ is a type A triple as depicted below.

$$\begin{array}{ccc} \boxed{\mathbf{a}} & \cdots & \boxed{\mathbf{b}} \\ \boxed{c} & & \boxed{d} \end{array}$$

The triple must be an inversion triple, so $I(F(a), F(b)) + I(F(b), F(c)) - I(F(a), F(c)) = 1$. We know that F contains no descents, so $I(F(a), F(c)) = 0$. Therefore $I(F(a), F(b)) + I(F(b), F(c)) = 1$ implies that either $F(a) > F(b)$ and $F(b) \leq F(c)$ or $F(a) \leq F(b)$ and $F(b) > F(c)$. If $F(a) > F(b)$ then $F(a) \neq F(b)$, so we need only consider the case where $F(a) \leq F(b)$. In this situation, since $F(b) > F(c)$, we must have $F(a) < F(b)$, for otherwise $F(a) = F(b) > F(c)$, which contradicts the fact that $F(a) \leq F(c)$.

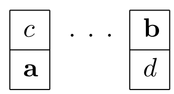
Next suppose $F(a) = F(b)$ and a and b are in the same row so that the column containing b is strictly taller than the column containing a. There must be a cell d on top of b such that $\{a, d, b\}$ is a type B triple as depicted below.



There are no descents in F, so $F(d) \leq F(b)$. Every triple of F is an inversion triple, so $I(F(d), F(a)) + I(F(a), F(b)) - I(F(d), F(b)) = 1$ implies that $I(F(d), F(a)) + I(F(a), F(b)) - 0 = 1$. We only need to consider the situation in which $F(a) \leq F(b)$, which implies that $I(F(a), F(b)) = 0$. This means that $I(F(d), F(a)) = 1$, so $F(d) > F(a)$. But $F(d) \leq F(b)$, so $F(a) < F(b)$. □

Lemma 3.3. *Let F be a descentless augmented filling such that every triple of F is an inversion triple. For each pair of cells a and b in F, with a to the left of b in the row directly below b, we have $F(a) \neq F(b)$.*

Proof. Consider two cells a and b in the augmented filling situated as described. There exists a cell d immediately below b and possibly a cell c immediately above a as depicted below.



If the column containing a is taller than or equal to the column containing b, then a lies directly below the cell c which must have $F(c) \leq F(a)$. Since the triple $\{c, b, a\}$ is a type A triple, it must satisfy $I(F(c), F(b)) + I(F(b), F(a)) - I(F(c), F(a)) = 1$. Since $F(c) \leq F(a)$, we have $I(F(c), F(a) = 0$ and hence $I(F(c), F(b)) + I(F(b), F(a)) = 1$. We only need to consider the situation in which $F(c) \leq F(b)$, which implies that $I(F(c), F(b)) = 0$. In this case, $I(F(b), F(a)) = 1$, so $F(b) > F(a) \geq F(c)$. This means that $F(b) > F(c)$ and hence $F(b) \neq F(c)$.

If the column containing b is strictly taller than the column containing a, the triple $\{a, b, d\}$ is a type B triple and must satisfy $I(F(b), F(a)) + I(F(a), F(d)) - I(F(b), F(d)) = 1$. There are no descents in F, so $F(b) \leq F(d)$ and $I(F(b), F(d) = 0$. Therefore either $I(F(b), F(a)) = 1$ or $I(F(a), F(d) = 1$. If $I(F(b), F(a)) = 1$, then $F(b) > F(a)$ and we are done. Therefore assume $I(F(b), F(a)) = 0$ and $I(F(a), F(d)) = 1$. This means that $F(a) > F(d) \geq F(b)$. So $F(a) > F(b)$ and hence $F(a) \neq F(b)$. □

Corollary 3.4. *The descent and inversion conditions used to describe the semi-skyline augmented fillings are enough to guarantee that the filling is non-attacking.*

Corollary 3.4 allows us to reformulate the combinatorial interpretation of $E_\gamma(X; 0, 0)$ as follows.

Definition 3.5. Let γ be a weak composition of n into m parts (where $m \in \mathbb{Z}^+ \cup \infty$). The polynomial $E_\gamma(X; 0, 0)$ in the variables $X = (x_1, x_2, ..., x_k)$ is the formal power series

$$E_\gamma(X; 0, 0) = \sum_{F \in SSAF(\gamma)} X^F,$$

where $SSAF(\gamma)$ is the set of all descent-less fillings of F in which every triple is an inversion triple.

Now that we have a description of Demazure atoms and a definition of the polynomials $E_\gamma(X; 0, 0)$, we are ready to understand their connection.

4. Proof of Theorem 1.1

The set of Demazure atoms for the partition λ can be considered as a decomposition of the Schur function s_λ. For any partition λ of n, we have ([**6**])

$$\sum_{\omega \in S_n} \mathfrak{U}(\omega, \lambda) = s_\lambda.$$

The functions $E_\gamma(X;0,0)$ are also a decomposition of the Schur functions [**7**], so it is natural to determine their relationship to the standard bases. Theorem 1.1 states that $\mathfrak{U}(\omega,\lambda) = NS_{\omega(\lambda)}$, where $\omega(\lambda)$ denotes the action of ω on the parts of λ.

4.1. Several Useful Lemmas. Consider a semi-standard Young tableau $T \in \mathfrak{U}(\omega,\lambda)$. Lascoux and Schützenberger's work implies that either $f_i(T) \in \mathfrak{U}(\omega,\lambda)$ or $f_i(T) \in \mathfrak{U}(s_i\omega,\lambda)$.

PROPOSITION 4.1. *There exists a map* Θ_i :*SSAF*$\longrightarrow$*SSAF such that the following diagram commutes for all SSYT T,*

$$\begin{array}{ccc} T & \xrightarrow{\ f_i\ } & T' \\ \Big\downarrow \Psi & & \Big\downarrow \Psi \\ F & \xrightarrow{\ \Theta_i\ } & F' \end{array}$$

where Ψ is the weight-preserving, shape permuting bijection between semi-standard Young tableaux and semi-skyline augmented fillings.

PROOF. Let F be an arbitrary semi-skyline augmented filling and let *read*(F) be the reading word obtained by reading F left to right, top to bottom. First match any pair i and $i+1$ which occur in the same row of F and remove these entries from the reading word of F. Next apply the parenthetical matching procedure on the reading word to determine which of the remaining occurrences of i and $i+1$ are unmatched.

Pick the rightmost unmatched i. Convert it to $i+1$. (If there is none, then $\Theta_i(F) = F$.) The result is a collection of rows which differ from *read*(F) in precisely one entry. Lemma Lemma 3.2 of [**7**] provides a procedure for mapping this collection of rows to a unique SAF. This SAF is $\Theta_i(F) = F'$. We must show that $\Theta_i(\Psi(T)) = \Psi(\tilde{\theta}_i(T))$.

Given an arbitrary SSYT T, consider the semi-skyline augmented filling $F = \Psi(T)$. We claim that sending the rightmost unmatched i in T to an $i+1$ affects the row of F which contains the rightmost unmatched i in *read*(F). To see this, begin with the word *col*(T) and let i_0 be the rightmost unmatched i in *col*(T). Let $i_1, i_2, \ldots, i_m$ be the matched i's appearing to the right of i_0, numbered from left to right.

There are at least k entries equal to $i+1$ between i_0 and i_k for each k such that $1 \le k \le m$. Suppose an entry i_k to the right of i_0 in *col*(T) is unmatched in F. Then at least one of the $(i+1)$'s, say $(i+1)_r$, which appeared to the right of i_0 must be matched to an $i = i_s$ weakly to the left of i_0. This can occur only if i_s appears in the same row as $(i+1)_r$ or after $(i+1)_r$ in *read*(F).

The entry $(i+1)_r$ is inserted before i_s, so during the insertion of some letter α the entry i_s must reach the cell immediately above $(i+1)_r$. Unless the entry on top of $(i+1)_r$ is greater than i, the entry i_s is placed on top of $(i+1)_r$. The only entry greater than i but less than or equal to $i+1$ is $i+1$. Therefore if i_s is not row-wise above $(i+1)_r$, then there exists another $i+1$ which could be matched to i_k. This argument can be carried out for each entry i weakly before i_0 in reading order, so i_k must remain matched in F. Therefore each entry i_k appearing after i_0 in *col*(T) remains matched in F.

The lowest candidate for a row containing an unmatched i in F is therefore the row containing i_0. However, it is possible that i_0 is matched in F to some $i+1 = (i+1)_2$ which was matched to some value $i_2 \neq i_0$ in *col*(T).

If this is the case, then $(i+1)_2$ passed i_2 in F during some insertion process without bumping i_2. Then i_2 is on top of some value less than $i+1$, and hence i_2 is on top of an entry i. Since $(i+1)_2$ passed every entry i between i_2 and i_0, the column containing i_2 contains only i's between the row containing i_2 and the row containing i_0. Therefore, $(i+1)_2$ is placed in the same row as i_0.

Suppose there are q such i's in the column containing i_2. Then there must be at most $q-1$ $(i+1)$'s which are matched to these i's in F, for otherwise there are q $(i+1)$'s to the right of $q-1$ matched i's and one unmatched i in *col*(T). If that were the case then one of the $(i+1)$'s would have been matched to i_0 in *col*(T). Therefore the number $(+1)$'s which are matched to the i's in the column containing i_2 is p, where $p < q$.

Therefore the p^{th} highest entry i in the column containing i_2 is the unmatched i in F which is converted to an $i+1$ during $\Theta_i(F)$. In $\tilde{\theta}_i(\Psi(T))$, the entry $(i+1)_0$ fills the position occupied by $(i+1)_2$ in $\Psi(T)$. Then the other values $i+1$ are shifted up by one row. Similarly, the second i in the column of containing i_2 now occupies the position filled by i_0 in $\Psi(T)$. So the i's are shifted down by one row. Hence the row which appears different in $\tilde{\theta}_i(\Psi(T))$ is the row which contains the p^{th} highest i of this column in $\Psi(T)$. This row contained an i in $\Psi(T)$ but contains an $i+1$ in $\tilde{\theta}_i(\Psi(T))$. But this is precisely the row which contained the rightmost unmatched i in $read(\Psi(T))$, and hence the row in which an i is converted to an $i+1$. So the rows affected by $\tilde{\theta}_i$ and Θ_i are the same.

The only way the rows of $\Psi(T)$ and $\Psi(\tilde{\theta}_i(T))$ could differ other than this one change would be if some letter α which is inserted on top of the new $i+1$ in $\Psi(\tilde{\theta}_i(T))$ cannot be placed on top of i in $\Psi(T)$. This could only happen if the letter α is equal to $i+1$. But if $i+1$ is inserted in the row directly above $i+1$ in $\Psi(\tilde{\theta}_i(T))$, then this $i+1$ would be matched in $\Psi(T)$ to the i which is changed to $i+1$. This contradicts the fact that the i converted to an $i+1$ is unmatched. Therefore the rows of $\Psi(\tilde{\theta}_i(T))$ are the same as the rows of $\Psi(T)$ except for the altering of the rightmost unmatched i.

The map Θ_i sends $\Psi(T)$ to the unique SAF whose rows are identical to those of $\Psi(T)$ other than the altering the same row which is altered in $\Theta_i(\Psi(T))$. Therefore $\Theta_i(\Psi(T)) = \Psi(\tilde{\theta}_i(T))$ and the diagram commutes. □

We need one additional Lemma to prove Theorem 1.1.

LEMMA 4.1. *For $F \in NS_\gamma$, either $\Theta_i(F) \in E_\gamma$ or $\Theta_i(F) \in E_{\sigma_i\gamma}$. In particular, $\Theta_i(F) \in E_{\sigma_i\gamma}$ precisely when each row below the rightmost unmatched i contains both an i and an $i+1$.*

PROOF. Assume that $F \in E_\gamma$. If $\Theta_i(F) = F$, then $\Theta_i(F) \in E_\gamma$. We must show that in the case where an unmatched i is sent to $i+1$, the resulting skyline augmented filling is either in E_γ or in $E_{\sigma_i\gamma}$. The map Θ_i affects the lowest (row-wise) unmatched i, denoted i_0, in F. Suppose there are no occurrences of the letter i in the row directly below the row r containing the converted i, denoted $(i+1)_0$ in $\Theta_i(F)$. Then $(i+1)_0$ is mapped to the same position in $\Theta_i(F)$ as i_0 in F. If there are no $i+1$'s in the row $r+1$ above r, then the entries in row $r+1$ are sent to the same positions as in F, and so the shape is still γ.

If there exists an $i+1 = (i+1)_{r+1}$ in row $r+1$, then there must be an i in row $r+1$, for otherwise $(i+1)_{r+1}$ would be matched to the i_0 in row r. If $(i+1)_{r+1}$ lies to the left of i_0 in F, then it is mapped to the same position in $\Theta_i(F)$ and everything remains the same. Otherwise, it will be placed on top of the $i+1 = (i+1)_0$ which replaced i_0. Then the $i = i_{r+1}$ in row $r+1$ will be placed in the position previously occupied by $i+1$. The only entries in row $r+2$ affected by this change are i and $i+1$. Again, if there is an $i+1 = (i+1)_{r+2}$ in row $r+2$ then there must be an $i = i_{r+2}$ in row $r+2$. By similar reasoning, the positions of the i and the $i+1$ will switch, but all other entries remain in the same position as in F. Therefore the heights of the columns containing i_0 and $(i+1)_0$ remain the same. If there are no occurrences of the entry i in row $r-1$, then the shape of the result is γ regardless of the entries in higher rows.

Next suppose there exists an $i = i_{r-1}$ in the row $r-1$ directly below i_0. There must also be an $i+1 = (i+1)_{r-1}$ in row $r-1$, since i_{r-1} is matched. If $(i+1)_{r-1}$ appears to the left of i_{r-1}, then i_0 could not have been on top of i_{r-1} in F, because i_0 would reach the empty cell on top of $(i+1)_{r-1}$ first and could not be bumped since there is no $i+1$ in row r of F. Therefore, $(i+1)_0$ is mapped to the same cell in $\Theta_i(F)$ as i_0 in F. In this case, we are back in the previous situation.

Finally assume that i_{r-1} appears to the left of $(i+1)_{r-1}$ in F. If the column containing $(i+1)_{r-1}$ is strictly taller than the column containing i_{r-1}, then the entry on top of $(i+1)_{r-1}$ must be equal to $i+1$ to ensure a type B inversion triple. But this contradicts the fact that there are no $i+1$ entries in row r.

Therefore the column containing i is weakly taller than the column containing $i+1$. The entry below i must be less than $i+1$ to ensure that these three cells are a type A inversion triple. This entry must also be greater than or equal to i to avoid descents. Hence it must be equal to i, denoted i_{r-2}. There must be an $i+1 = (i+1)_{r-2}$ in row $r-2$, since i_{r-2} is matched in F. Since F is a non-attacking filling, $(i+1)_{r-2}$ must appear weakly to the right of $(i+1)_{r-1}$. The entry α immediately below $(i+1)_{r-1}$ must have $\alpha \geq i+1$. The column containing $(i+1)_{r-1}$ must be weakly taller than the column containing $(i+1)_{r-2}$ if the columns are distinct. This implies that the column containing i_{r-2} is weakly taller than the column containing $(i+1)_{r-2}$. This again implies that the entry below i_{r-2} is i. Continuing in this manner, we have a column consisting only of i's, so this must be the i^{th} column. Since there must be an $i+1$ in each of these rows, there must be an $i+1$ in the first row. Any entry in the bottom row lies on top of its corresponding entry in the basement row. Therefore the $i+1$ in the bottom row is in the $(i+1)^{th}$ column. This entry is weakly to the right of the other $(i+1)$'s. Since it is immediately to the right of the i^{th} column, this means that the $(i+1)^{th}$ column contains only the entries $i+1$ until the r^{th} row.

The entry i_0 cannot lie strictly to the right of i_{r-1} since F is non-attacking. But it cannot lie strictly to the left of the i^{th} column without creating a descent. Therefore i_0 is in the i^{th} column. Sending i_0 to $i+1=(i+1)_0$ moves it from the i^{th} column in F to the $(i+1)^{th}$ column in $\Theta_i(F)$. If an entry appears in row r of F on top of $(i+1)_{r-1}$, then this entry appears on top of i_{r-1} in $\Theta_i(F)$. The other entries in row r appear in the same positions in F and $\Theta_i(F)$.

There are no unmatched $(i+1)$'s in the row $r+1$ above r, since they would be matched to i_0. If there is an $i=i_{r+1}$ in the row above r and no $i+1$, then i_{r+1} appears on top of $(i+1)_0$ in $\Theta_i(F)$ and on top of i_0 in F. Any entry which appears on top of the cell in row r of column $i+1$ of F appears in row r of column i of $\Theta_i(F)$. The remaining entries in row r appear in same positions in F and $\Theta_i(F)$.

If both i and $i+1$ appear in row $r+1$ of F, the entry $i+1=(i+1)_{r+1}$ is placed on top of $(i+1)_0$ and i_{r+1} is placed in the position occupied by $(i+1)_{r+1}$ in F. The remaining entries appear in the same position in F and $\Theta_i(F)$.

If neither i nor $i+1$ appears in row $r+1$ of F, then the rest of the column containing $(i+1)_0$ is the same as the rest of the i_0 column in F. The other entries of row $r+1$ appear in the same position in F and $\Theta_i(F)$.

In each of the above cases, the only columns which differ are the i^{th} column and the $(i+1)^{th}$ column. These columns are permuted in $\Theta_i(F)$. Therefore Θ_i permutes the i^{th} and $(i+1)^{th}$ column, resulting in shape $\sigma_i\gamma$. This case occurs only when columns i and $i+1$ contain only the entries i and $i+1$ respectively. □

4.2. Proof of Theorem 1.1. We are now ready to prove that the Demazure atoms are equivalent to the functions $E_\gamma(X;0,0)$.

PROOF. Fix a partition λ and argue by induction on the length of the permutation ω in $\mathfrak{U}(\omega,\lambda)$. First let ω be the identity. Then $\mathfrak{U}(\omega,\lambda)$ is the dominant monomial. Consider λ as a composition of n into n parts by adding zeros to the right if necessary. Each cell a in λ_1 must have $F(a)=1$, for otherwise there would be a descent. If the second column contained a cell b such that $F(b)=1$, this cell and the cell beside it in first column would be attacking. To avoid descents, each cell b in the second column must have $F(b)\le 2$. Therefore each cell b in the second column has $F(b)=2$. Continuing inductively, we see that each cell c in the i^{th} column must have $F(c)=i$. To see that this is indeed an SAF, we need only to check type A triples. But if the two cells in the left-hand column are equal and less than the cell in the right-hand column, the result is a type A inversion triple. Therefore, the $E_\lambda=\mathfrak{U}(id,\lambda)$.

Next assume that $\mathfrak{U}(\omega,\lambda)=E_{\omega(\lambda)}$, where $\omega(\lambda)$ is the permutation ω applied to the columns of λ when λ is considered as a composition of n into n parts. The monomials in $\mathfrak{U}(\sigma_i\omega,\lambda)$ are the images of monomials of $\mathfrak{U}(\omega,\lambda)$ whose image under (possibly multiple applications of) $\tilde{\theta}_i$ is not a monomial of $\mathfrak{U}(\omega,\lambda)$. Pick some such monomial of $\mathfrak{U}(\omega,\lambda)$ which maps to $\mathfrak{U}(\sigma_i\omega,\lambda)$, represented by the SSYT T such that $\tilde{\Theta}_i(T)\in\mathfrak{U}(\sigma_i\omega,\lambda)$. By Proposition 4.1, $\Psi(\tilde{\theta}_i(T))=\Theta_i(\Psi(T))$. Since $\Psi(T)\in E_{\omega(\lambda)}$, Lemma 4.1 implies that $\Theta_i(\Psi(T))\in E_{\omega(\lambda)}$ or $\Theta_i(\Psi(T))\in E_{\sigma_i\omega(\lambda)}$. If $\Theta_i(\Psi(T))\in E_{\omega(\lambda)}$, then $\Psi(\tilde{\theta}_i(T))\in E_{\omega(\lambda)}$, so $\tilde{\theta}_i(T)\in\mathfrak{U}(\omega,\lambda)$ because $\mathfrak{U}(\omega,\lambda)=E_{\omega(\lambda)}$ by the inductive hypothesis. This contradicts the assumption that $\tilde{\theta}_i(T)\in\mathfrak{U}(\sigma_i\omega,\lambda)$, so $\Theta_i(\Psi(T))\in E_{\sigma_i\omega(\lambda)}$. Therefore, $\mathfrak{U}(\sigma_i\omega,\lambda)\subseteq E_{\sigma_i\omega(\lambda)}$.

Let F be a monomial in $E_{\sigma_i\omega(\lambda)}$. Consider *read*$(F)$. Match the occurrences i and $i+1$ of F as in Proposition 4.1 above. Find the leftmost unmatched $i+1=(i+1)_0$. Send it to i and map the row entries to an SSAF according to the procedure in Lemma **??**. The resulting reading word is the reading word of an SSAF F' where this $i=i_0$ is the rightmost unmatched i. (If there were any unmatched values of i to the right of i_0 in *read*(F), they would have been matched to $(i+1)_0$ in *read*(F). If i_0 were matched to some $i+1$ to its left, this $i+1$ would be an unmatched $i+1$ in *read*(F) farther left than the selected $i+1$.) Therefore $\Theta_i(F')=F$. If $F'\in E_{\omega(\lambda)}$, then $\Psi^{-1}(F)=T$ is the semi-standard Young tableau in $\mathfrak{U}(\omega,\lambda)$ such that $\tilde{\theta}_i(T)$ maps to F. Otherwise, repeat the procedure on F' until an element of $E_{\omega(\lambda)}$ is reached. This happens eventually since by Lemma 4.1, an element of $E_{\omega(\lambda)}$ is sent by Θ_i to an element of $E_{\sigma_i\omega(\lambda)}$ precisely when there are no unmatched $(i+1)$s to the right of the rightmost unmatched i in *read*(F). F' satisfies this condition when the leftmost unmatched $i+1$ in F is also the rightmost unmatched $i+1$. Since the number of i's in F must be less than or equal to the number of $(i+1)$'s, this will eventually occur. Therefore $E_{\sigma_i\omega(\lambda)}\subseteq\mathfrak{U}(\sigma_i\omega,\lambda)$. So $E_{\sigma_i\omega(\lambda)}=\mathfrak{U}(\sigma_i\omega,\lambda)$. □

Theorem 1.1 provides a non-inductive construction of the standard bases. In particular, given a partition $\lambda\vdash n$ and a permutation $\omega\in S_n$, first consider λ as a composition of n into n parts by adding zeros to the right if necessary. Then apply the permutation ω to the columns of I to get the shape $\omega(\lambda)$. Finally, determine all skyline augmented fillings of the shape $\omega(\lambda)$. The monomials given by the weights of these SSAFs are the monomials of $\mathfrak{U}(\omega, \lambda)$.

5. Computation of right keys

Recall that the standard basis $\mathfrak{U}(\omega, \lambda)$ is equal to the sum of the weights of all SSYT with right key $K(\omega, \lambda)$. Therefore all of the SSYT which map to an SSAF of shape $\omega(\lambda)$ have the same right key, $K(\omega, \lambda)$. The *super SSAF* (denoted $super(\gamma)$) of a composition γ is the SSAF of shape γ whose i^{th} column contains only the entries i. Theorem 1.1 provides a simple method to determine the right key of a semi-standard Young tableau.

Corollary 5.1. *Given an arbitrary SSYT T, let γ be the shape of $\Psi(T)$. Then $K_+(T) = key(\gamma)$.*

Proof. We must show that the map $\Psi : SSYT \to SSAF$ sends a key T to $super(\gamma)$, where γ is the composition $content(T)$. Prove this by induction on the number of columns of T. If T has only one column, $C_1 = \alpha_1 \ \alpha_2 \ \dots \alpha_j$, then this column maps to a filling F with one row such that the α_i^{th} column contains the entry α_i. This is precisely $super(content(T))$.

Next, assume that $\Psi(T) = super(content(T))$ for all keys T with less than or equal to $m-1$ columns. Let S be a key with m columns. After the insertion of the rightmost $m-1$ columns, the figure is $super(content(S \setminus C_1))$ by the inductive hypothesis. We must show that inserting the first column produces $super(content(S))$.

Let $C_1 = \alpha_1 \ \alpha_2 \ \dots \alpha_l$. The last entry, α_1 to be inserted, must be inserted on top of the α_1^{th} column, since all other columns contain strictly smaller entries. Therefore if α_2 is inserted here during a previous step, α_2 is bumped to the α_2^{th} column. If not, then α_2 was already on top of the α_2^{th} column, since all other columns contain entries which are strictly smaller than α_2. Continuing in this manner we see that after all elements of C_1 have been inserted, the element α_i must lie on top of the α_i^{th} column. Therefore $\Psi(S) = super(content(S))$.

Since $super(\omega(\lambda)) \in \mathfrak{U}(\omega, \lambda)$, each SSYT which maps to a SSAF of shape $\omega(\lambda)$ has the same right key as $super(\omega(\lambda))$. Therefore, if $T \in \mathfrak{U}(\omega, \lambda)$, then $key(T) = key(\omega(\lambda))$. □

This is a simple and quick procedure for calculating the right key of any SSYT which facilitates the computation of Demazure atoms and Demazure characters. In particular, let γ be a composition which rearranges a partition λ. The *key polynomial* κ_γ is given by the weights of all semi-standard Young tableaux T of shape λ such that $K_+(T) \leq key(\gamma)$. Therefore

$$\kappa_\gamma = \sum_{\alpha \geq \gamma} E_\alpha,$$

where the order is the Bruhat order on compositions.

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Department of Mathematics, University of California, Berkeley, CA
E-mail address: sarahm@math.berkeley.edu
URL: http://math.berkeley.edu/~sarahm