Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
Nankai University, Tianjin, China 2007

Alternating subgroups of Coxeter groups (Extended Abstract)

Francesco Brenti, Victor Reiner, and Yuval Roichman

Abstract. We explore combinatorial consequences of a simple presentation due to Bourbaki for the alternating subgroup of a Coxeter group.

1. Introduction

For any Coxeter system (W, S), its *alternating subgroup* W^+ is the kernel of the *sign character* that sends every $s \in S$ to -1. An exercise from Bourbaki gives a simple presentation for W^+, after one distinguishes a generator $s_0 \in S$. The goal here is to explore the combinatorial properties of this presentation, at four different levels of generality (defined below) regarding the generator s_0:

s_0 arbitrary
(*Section* 2)
/ \
s_0 evenly-laced (*Section* 3) — s_0 a leaf (*Section* 4)
\ /
s_0 an even leaf
(*Section* 5)

Section 2 reviews the presentation and explores some of its consequences in general for the *length function*, *parabolic subgroups*, a *Coxeter-like complex* for W^+, and the notion of *palindromes*, which play the role usually played by *reflections* in a Coxeter system. This section also defines *weak* and *strong* partial orders on W^+ and poses some basic questions about them.

Section 3 explores the special case where s_0 is *evenly-laced*, meaning that m_{0i} is even for all i. It turns out that, surprisingly, this case is much better-behaved. Here the unique, length-additive factorization $W = W^J W_J$ for parabolic subgroups of W induces similar unique length-additive factorizations within W^+.

2000 *Mathematics Subject Classification.* 20F55,20F05.

Key words and phrases. Coxeter group, alternating group, presentation, length, Poincaré series.

Second author supported by NSF grant DMS-0245379. Third author supported in part by the Israel Science Foundation, founded by the Israel Academy of Sciences and Humanities.

One can easily compute generating functions for W^+ by length, or jointly by length and certain descent statistics. Here the palindromes which shorten an element determine that element uniquely, and satisfy a crucial *strong exchange property.* This gives better characterizations of the weak and strong partial orders, and resolves affirmatively all the questions about the partial orders from Section 2 in this case.

Section 4 examines how the general presentation simplifies to what we call a *nearly Coxeter* presentation when s_0 is a *leaf* in the Coxeter diagram, meaning that s_0 commutes with all but one of the other generators $S - \{s_0\}$. Such leaf generators occur in many situations, e.g. when W is finite[1] and for most affine W.

Section 5 explores the further special case where s_0 is an evenly-laced leaf node, that is, both a leaf and evenly-laced. The classification of finite and affine Coxeter systems shows that *all* evenly-laced nodes s_0 are even leaves when W is finite, and this is almost always the case for W affine. In particular, even leaf nodes occur in the finite type $B_n = (C_n)$ and the affine types $\tilde{B}_n, \tilde{C}_n$. When s_0 is an even leaf, there is an amazingly close connection between the alternating group W^+ and a *different* index 2 subgroup W', namely the kernel of the homomorphism χ_0 sending s_0 to -1 and all other Coxeter generators to $+1$. It turns out that this subgroup W' is a (non-parabolic) reflection subgroup of W, carrying its own Coxeter presentation (W', S'), closely related to the presentation for (W, S). This generalizes the inclusion of type $W' = W(D_n)$ inside $W = W(B_n)$, and although $W^+ \not\cong W'$, the connection allows one to reduce all of the various combinatorial aspects of the presentation (W^+, R) (length function, descent sets, partial orderings, reduced words) to their well-studied analogues for the Coxeter system (W', S').

This is an extended abstract. Proofs and more details are given in [**6**].

2. The general case

2.1. Bourbaki's presentation. Let (W, S) be a Coxeter system with generators $S = \{s_0, s_1, \ldots, s_n\}$, that is, W has a presentation of the form

$$W = \langle S = \{s_0, s_1, \ldots, s_n\} : (s_i s_j)^{m_{ij}} = e \text{ for } 0 \le i \le j \le n \rangle \tag{1}$$

where $m_{ij} = m_{ji} \in \{2, 3, \ldots\} \cup \{\infty\}$ and $m_{ii} = 2$.

The *sign character* $\epsilon : W \to \{\pm 1\}$ is the homomorphism uniquely defined by $\epsilon(s) = -1$ for all $s \in S$. Its kernel $W^+ := \ker(\epsilon)$ is an index two subgroup called the *alternating subgroup* of W.

Once one has distinguished s_0 in S by its zero subscript, an exercise in Bourbaki [**5**, Chap. IV, Sec. 1, Exer. 9] suggests a simple presentation for W^+, which we recall here.

PROPOSITION 2.1.1. *Given a Coxeter system (W, S) with distinguished generator s_0, map the set $R = \{r_1, \ldots, r_n\}_{i=1,2,\ldots,n}$ into W^+ via $r_i \mapsto s_0 s_i$. Then this gives a set of generators for W^+ with the following presentation:*

$$\begin{aligned} W^+ \cong \langle R = \{r_1, \ldots, r_n\} :& \\ & r_i^{m_{0i}} = (r_i^{-1} r_j)^{m_{ij}} = e \textit{ for } 1 \le i < j \le n \rangle. \end{aligned} \tag{2}$$

---

[1] Combinatorial aspects of this nearly Coxeter presentation were explored for W of type A in [**12**], and partly motivated the current work.

2.2. Length with respect to $R \cup R^{-1}$. Given a group G and a generating subset $A \subset G$, let A^* denote the set of all words $\mathbf{a} = (a_1, \dots, a_\ell)$ with letters a_i in A. Let $A^{-1} := \{a^{-1} : a \in A\}$. Let $\ell_A(\cdot)$ denote the length function on G with respect to the set A, that is, $\ell_A(g) := \min\{\ell : g = a_1 a_2 \cdots a_\ell \text{ for some } a_i \in A\}$. Given an A^*-word $\mathbf{a}$ that factors g in G, say that $\mathbf{a}$ is a *reduced word* for g if it achieves the minimum possible length $\ell_A(g)$.

Definition 2.2.1. Given a Coxeter system (W, S) with $S = \{s_0, s_1, \dots, s_n\}$ as before, let $\nu(w)$ denote the minimum number of generators $s_j \neq s_0$ occurring in any expression $\mathbf{s} = (s_{i_1}, \cdots, s_{i_\ell}) \in S^*$ that factors w in W, i.e. $w = s_{i_1} \cdots s_{i_\ell}$.

Proposition 2.2.2. *For a Coxeter system (W, S) with $S = \{s_0, s_1, \dots, s_n\}$ as before, and the presentation (W^+, R) in (2), one has*

$$\ell_{R \cup R^{-1}}(w) = \nu(w)$$

for all $w \in W^+$.

Example 2.2.3. Let (W, S) be the symmetric group $W = \mathfrak{S}_n$, with $S = \{s_0, s_1, \dots, s_{n-2}\}$ in which s_i is the adjacent transposition $(i+1, i+2)$, so $s_0 = (1, 2)$; this is the usual Coxeter system of type A_{n-1}. Given a permutation $w \in \mathfrak{S}_n$, let $\mathrm{lrmin}\,(w)$ denote its number of *left-to-right minima*, that is, the number of $j \in \{2, 3, \dots, n\}$ satisfying $w(i) > w(j)$ for $1 \leq i < j$. Let $\mathrm{inv}(w)$ denote its number of *inversions*, that is, the number of pairs (i, j) with $1 \leq i < j \leq n$ and $w(i) > w(j)$.

Proposition 2.2.4. *For any even permutation $w \in \mathfrak{S}_n$*

$$\ell_{R \cup R^{-1}}(w) = \mathrm{inv}(w) - \mathrm{lrmin}\,(w).$$

In [**12**] it was shown for (W, S) of type A_{n-1} with s_0 a leaf node as above, one has

$$\sum_{w \in W^+} q^{\ell_{R \cup R^{-1}}(w)} = (1 + 2q)(1 + q + 2q^2) \cdots (1 + q + q^2 \cdots + q^{n-3} + 2q^{n-2}). \tag{3}$$

For refinements that incorporate other statistics see [**12**, Prop. 5.7(2), 5.11(2)].

2.3. Parabolic subgroup structure for (W^+, R). The structure of parabolic subgroups W_J for (W, S) is an important part of the theory. For (W^+, R) one finds that its parabolic subgroups are closely tied to the parabolic subgroups W_J containing s_0.

Definition 2.3.1. For any $J \subset R = \{r_1, \dots, r_n\}$, the subgroup $W_J^+ = \langle J \rangle$ generated by J inside W^+ will be called a *(standard) parabolic subgroup.*

Proposition 2.3.2. *Define $\tau : W \to W^+$ by*

$$\tau(w) = \begin{cases} w & \text{if } w \in W^+ \\ w s_0 & \text{if } w \notin W^+ \end{cases}$$

Then for any $J \subseteq S$ with $s_0 \in J$, one has $W_J \cap W^+ = W^+_{\tau(J)}$.

Also, the (set) map τ induces a W^+-equivariant bijection

$$W/W_J \xrightarrow{\tau} W^+/W^+_{\tau(J)}.$$

Furthermore, the coset representatives $\tau(W^J)$ for $W^+/W^+_{\tau(J)}$ each achieve the minimum $\ell_{R \cup R^{-1}}$-length within their coset.

Note that, in general, an element of $\tau(W^J)$ is not *unique* in achieving the minimum length $\ell_{R\cup R^{-1}}$ within its coset, unless s_0 is evenly-laced; see Subsection 3.2 below.

2.4. The Coxeter complex for (W^+, R)**.** The results of Section 2.3 allow us to define a Coxeter complex[2] $\Delta(W^+, R)$, and the map τ allows one to immediately carry over many of the properties of $\Delta(W, S)$.

DEFINITION 2.4.1. Given a Coxeter system (W, S) with $S = \{s_0, s_1, \ldots, s_n\}$, and the ensuing presentation (2) for W^+ via the generators $R = \{r_1, \ldots, r_n\}$, define the *Coxeter complex* to be the simplicial complex $\Delta(W^+, R)$ which is the nerve of the covering of the set W^+ by the maximal (proper) parabolic subgroups

$$\{wW^+_{R\setminus\{r\}}\}_{w\in W^+, r\in R}.$$

PROPOSITION 2.4.2. *The Coxeter complex $\Delta(W^+, R)$ is W^+-equvariantly isomorphic, via the map τ, to the type-selected subcomplex $\Delta(W, S)_{S\setminus\{s_0\}}$, obtained by deleting all vertices of color s_0 from $\Delta(W, S)$. Consequently $\Delta(W^+, R)$ is a pure $(n-1)$-dimensional shellable simplicial complex, which is balanced with color set R.*

Similarly for any $J \subseteq R$, its type-selected subcomplex $\Delta(W^+, R)_J$ is W^+-equivariantly isomorphic to the type-selected subcomplex $\Delta(W, S)_{\tau^{-1}(J)}$, where here $\tau^{-1}(J) := \{s_0\} \cup \{s_i \in S : r_i \in R\}$.

This has consequences for the (reduced) homology $\tilde{H}_*(\Delta(W^+, R), \mathbb{Z})$. Let $\mathbb{Z}[W/W_{S-\{s_0\}}]$ denote the permutation action of W^+ on cosets of the maximal parabolic $W_{S-\{s_0\}}$, or in other words,

$$\mathbb{Z}[W/W_{S-\{s_0\}}] = \mathrm{Res}^W_{W^+} \mathrm{Ind}^W_{W_{S-\{s_0\}}} \mathbf{1}.$$

If W is finite, denote by $\mathbb{Z}v$ the unique copy of the trivial representation contained inside $\mathbb{Z}[W/W_{S-\{s_0\}}]$, spanned by the sum v of all cosets $wW_{S-\{s_0\}}$.

COROLLARY 2.4.3. *The homology $\tilde{H}_*(\Delta(W^+, R), \mathbb{Z})$ is concentrated in dimension $n-1$. As a representation of W^+, it is the restriction from W of the representation on the top homology of $\Delta(W, S)_{S\setminus\{s_0\}}$. More concretely,*

$$H_*(\Delta(W^+, R), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[W/W_{S-\{s_0\}}] & \text{when } W \text{ is infinite,} \\ \mathbb{Z}[W/W_{S-\{s_0\}}]/\mathbb{Z}v & \text{when } W \text{ is finite.} \end{cases}$$

EXAMPLE 2.4.4. Let (W, S) be of type A_3, so that $W = \mathfrak{S}_3$, having Coxeter diagram which is a path with three nodes. If one labels the $S = \{s_0, s_1, s_2\} = \{(1, 2), (2, 3), (3, 4)\}$, so that s_0 is a leaf node in the Coxeter diagram, then the Figure 2.4.4(a) shows the Coxeter complex $\Delta(W^+, R)$ with facets labelled by W^+. Figure 2.4.4(b) shows the isomorphic type-selected subcomplex $\Delta(W, S)_{S-\{s_0\}}$ with facets labelled by $W^{\{s_0\}}$. Figure 2.4.4(c) shows the resulting Coxeter complex $\Delta(W^+, R)$ with facets labelled by W^+ after one relabels $S = \{s_0, s_1, s_2\} = \{(2, 3), (1, 2), (3, 4)\}$. so that now s_0 is the central node, not a leaf, and s_1, s_2 commute.

2.5. Palindromes versus reflections. For a Coxeter system (W, S), the set of reflections $\mathtt{T} := \bigcup_{w\in W, s\in S} wsw^{-1}$ plays an important role in the theory. A similar role for (W^+, R) is played by the set of *palindromes*, particularly when s_0 is evenly-laced. Palindromes will also give the correct way to define the analogues of the strong Bruhat order defined in Subsection 2.6 below.

---

[2] Actually, this is what was called a *Coxeter-like complex* for the presentation of W^+ by the generating set R in [**1**].

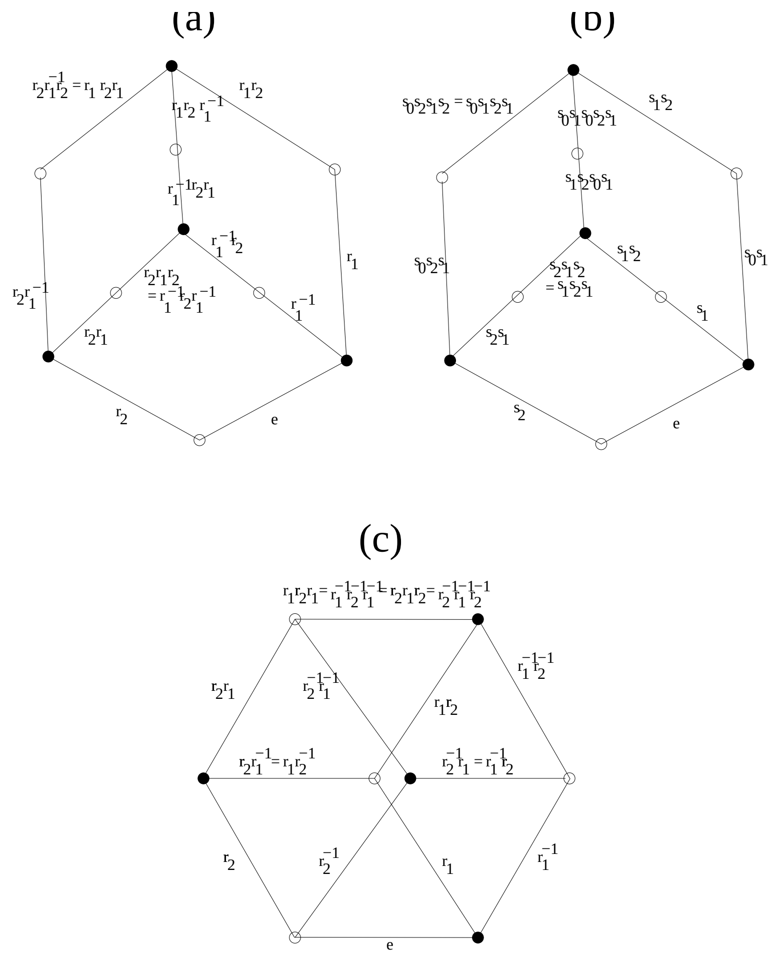


FIGURE 1. Coxeter complexes for (W^+, R) with (W, S) of type A_3. Figure (a) shows $\Delta(W^+, R)$ when s_0 is a leaf node, that is, the Coxeter diagram is labelled $s_0 - s_1 - s_2$, while (b) shows the isomorphic complex $\Delta(W, S)_{S-\{s_0\}}$. Figure (c) shows $\Delta(W^+, R)$ when s_0 is the non-leaf node, that is, the Coxeter diagram is labelled $s_1 - s_0 - s_2$.

DEFINITION 2.5.1. Given a pair (G, A) where G is a group generated by a set A, say that an element g in G is an *(odd) palindrome* if there is an $(A \cup A^{-1})^*$-word $\mathbf{a} = (a_1, \ldots, a_\ell)$ factoring g with ℓ odd and $a_{\ell+1-i} = a_i$ for all i. Denote the set of (odd) palindromes in G by $\mathtt{P}(G)$.

Let $\hat{\mathtt{T}} := \bigcup_{w \in W, s \in S \setminus \{s_0\}} wsw^{-1}$ denote the set of reflections in W that are conjugate to at least one $s \neq s_0$.

PROPOSITION 2.5.2. *For any Coxeter system* (W, S)*, one has*

$$\mathtt{P}(W^+)s_0 = \hat{\mathtt{T}} = s_0\mathtt{P}(W^+).$$

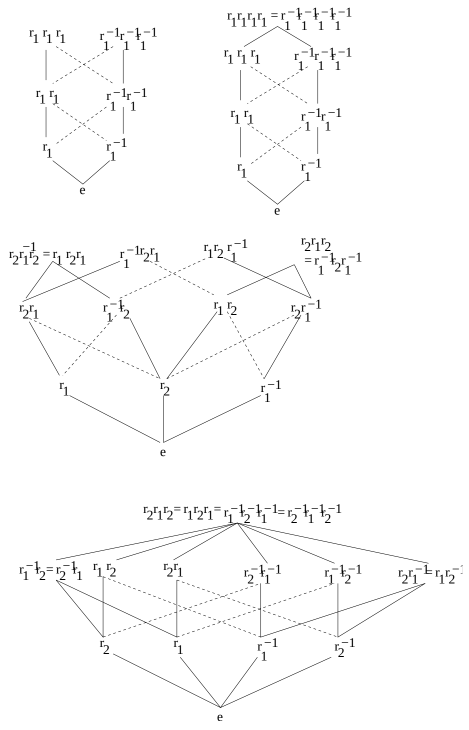


Figure 2. Examples of the left weak (solid edges) and left strong orders (solid and dotted edges) on W^+ for $(W,S) = I_2(7), I_2(8)$, and A_3 with s_0 labelling a leaf node versus a non-leaf node.

2.6. Weak and strong orders. For a Coxeter system (W,S) there are two related partial orders (the weak and strong orders) on W which form graded posets with rank function ℓ_S. Here we define analogues for (W^+, R).

Definition 2.6.1. Define the *(left) strong order* $<_{LS}$ on W^+ as the transitive closure of the relation $w \xrightarrow{p} pw$ if $p \in \mathsf{P}(W^+)$ and $\ell_{R\cup R^{-1}}(w) < \ell_{R\cup R^{-1}}(pw)$. Similarly define the *(right) strong order* $<_{RS}$.

Define the *(left) weak order* $<_{RW}$ on W^+ as the transitive closure of the relation $w \lessdot_{RW} wr$ if $r \in R \cup R^{-1}$ and $\ell_{R\cup R^{-1}}(w) + 1 = \ell_{R\cup R^{-1}}(rw)$. Similarly define the *(right) weak order* $<_{RS}$.

Figure 2.6 shows the left weak and left strong orders on W^+ for the two dihedral Coxeter systems $I_2(7), I_2(8)$, as well as for type A_3 with the two different choices for the node labelled s_0, as in Figure 2.4.4.

A glance at these figures, along with the good properties known for the usual weak and strong orders on (W,S), raise several obvious questions.

Question 2.6.2. *Are all of these orders graded by the function $\ell_{R\cup R^{-1}}$, that is, do all maximal chains have the same length?*

Question 2.6.3. *Do the weak orders form a meet semilattice in general?*

Question 2.6.4. *Is the strong order shellable?*

We will see in Subsection 3.5 that the answers to all of these questions are affirmative when s_0 is evenly-laced. Furthermore, in Section 5 it will be shown that when s_0 is an evenly-laced leaf node, the strong and weak orders coincide with the usual Coxeter group strong and weak orders for the related Coxeter system (W', S') defined there.

Remark 2.6.5. Some things are clearly *not* true of the various orders, even in the best possible situation where s_0 is an even leaf. Although the left weak/strong orders are isomorphic to the right weak/strong orders, they are not the *same* orders. Also, none of the four orders (left/right weak/strong) on W^+ coincides with the restriction from W to W^+ of the analogous left/right weak/strong order on W. Similarly, none of the four orders on W^+ coincides via the bijection τ with the restriction from W to $W^{\{s_0\}}$ of the analogous order on W.

3. The case of an evenly-laced node

When the distinguished generator s_0 in $S = \{s_0, s_1, \ldots, s_n\}$ for the Coxeter system (W, S) has the extra property that m_{0i} is even for $i = 1, 2, \ldots, n$, say that s_0 is an *evenly-laced node* of the Coxeter diagram. This has many good consequences for the presentation (W^+, R) explored in the next few subsections:

- the length function $\ell_{R \cup R^{-1}}$ simplifies,
- the coset representatives $\tau(W^J)$ for $W^+/W_{\tau(J)}$ from Section 2.3 are distinguished by their minimum length within the coset, and the length is additive in the decomposition $W^+ = \tau(W^J)\ W^+_{\tau(J)}$,
- the palindromes $\mathtt{P}(W^+)$ behave more like reflections, satisfying a *strong exchange condition*, and consequently
- the partial orders considered earlier are as well-behaved as their analogues for (W, S).

3.1. Length generating function.

It follows from Tits' solution to the word problem that when s_0 is evenly-laced, the number of occurrences of s_0 in *any* reduced word is the same; denote this quantity $\ell_0(w)$ and define $W(S; q_0, q) := \sum_{w \in W} q_0^{\ell_0(w)} q^{\nu(w)}$. The usual diagram-recursion methods [**9**, §5.12] for writing down the Poincaré series of W as a rational function in q generalize to compute the finer Poincaré series $W(S; q_0, q)$ [**10, 13**].

Definition 3.1.1. Define the $\ell_{R \cup R^{-1}}$ length generating function on W^+:

$$W^+(R \cup R^{-1}; q) := \sum_{w \in W^+} q^{\ell_{R \cup R^{-1}}(w)}.$$

Proposition 3.1.2. *When s_0 is evenly-laced,*

$$W^+(R \cup R^{-1}; q) = \left[\frac{W(S; q_0, q)}{1 + q_0} \right]_{q_0 = 1}.$$

Example 3.1.3.
Let (W, S) be the Coxeter system of type $B_n (= C_n)$, the group of *signed permutations* acting on $\mathbb{R}^n$. Index $S = \{s_0, s_1, \ldots, s_{n-1}\}$ so that s_0 is the special generator that negates the first coordinate, and s_i swaps the $i^{th}, (i+1)^{st}$ coordinates when $i \geq 1$. Then

$$W^+(R \cup R^{-1}; q) = [n]_q \left([2]_q [4]_q [6]_q \cdots [2n-4]_q [2n-2]_q \right),$$

where $[n]_q := \frac{1-q^n}{1-q}$.

3.2. Parabolic coset representatives revisited.

PROPOSITION 3.2.1. *When* s_0 *is evenly-laced, the coset representatives* $\tau(W^J)$ *for* $W^+/W^+_{\tau(J)}$ *are the unique representatives within each coset* $wW^+_{\tau(J)}$ *achieving the minimum* $\ell_{R\cup R^{-1}}$*-length.*

The assumption that m_{01} is even turns out to be crucial; see [**6**].

3.3. Descent statistics. For a Coxeter system (W,S), aside from the length statistic $\ell_S(w)$ for $w \in W$, one often considers the *descent set* and *descent number* of w defined by

$$\begin{aligned}\mathrm{Des}_S(w) &:= \{s \in S : \ell_S(ws) < \ell_S(w)\}\\ \mathrm{des}_S(w) &:= |\mathrm{Des}_S(w)|.\end{aligned}$$

Generating functions counting W jointly by ℓ_S and $\mathrm{Des}_S(w)$ are discussed in [**13**].

DEFINITION 3.3.1. Given $w \in W^+$, define its *weak descent* (or *nonascent*) *set* $\mathrm{Nasc}_{R\cup R^{-1}}(w)$ and its *symmetrized weak descent (nonascent) set* $\widehat{\mathrm{Nasc}}_R(w)$ as follows:

$$\begin{aligned}\mathrm{Nasc}_{R\cup R^{-1}}(w) &:= \{r \in R \cup R^{-1} : \ell_{R\cup R^{-1}}(wr) \leq \ell_{R\cup R^{-1}}(w)\} \subseteq R \cup R^{-1}\\ \widehat{\mathrm{Nasc}}_R(w) &:= \{r \in R : \text{ either } r \text{ or } r^{-1} \in \mathrm{Nasc}_{R\cup R^{-1}}(w)\} \subseteq R\end{aligned}$$

It turns out that nonascents in (W^+,R) relate to descents in (W,S) of the minimum length parabolic coset representatives $W^{\{s_0\}}$ for $W/W_{\{s_0\}}$. This is mediated by the inverse τ^{-1} to the bijection $\tau : W^{\{s_0\}} \to W^+$ that comes from taking $J = \{s_0\}$ in Proposition 2.3.2.

PROPOSITION 3.3.2. *For any Coxeter system* (W,S) *and* $s_0 \in S$ *and* $w \in W^+$*, one has an inclusion*

$$\widehat{\mathrm{Nasc}}_R(w) \supseteq \mathrm{Des}_S(\tau^{-1}(w)).$$

after identifying $R = \{r_1,\ldots,r_n\}$ *and* $S \setminus \{s_0\} = \{s_1,\ldots,s_n\}$ *with their subscripts* $[n] := \{1,2,\ldots,n\}$*. In general, this inclusion can be proper, but when* s_0 *is evenly-laced it is an equality:*

$$\widehat{\mathrm{Nasc}}_R(w) = \mathrm{Des}_S(\tau^{-1}(w)).$$

COROLLARY 3.3.3. *When* s_0 *is evenly-laced,*

$$\sum_{w\in W^+} \mathbf{t}^{\widehat{\mathrm{Nasc}}_{R\cup R^{-1}}(w)} q^{\ell_{R\cup R^{-1}}(w)} = \left[\sum_{w\in W} \mathbf{t}^{\mathrm{Des}_S(w)} q_0^{\ell_0(w)} q^{\nu(w)}\right]_{q_0=1,t_0=0}$$

where $\mathbf{t}^A := \prod_{j\in A} t_j$*.*

This last generating function for W is easily computed using the techniques from [**13**].

EXAMPLE 3.3.4. Consider the Coxeter system (W,S) of type B_n, labelled as in Example 3.1.3. Then [**13**, §II, Theorem 3] shows that

$$\sum_{w\in W} \mathbf{t}^{\mathrm{Des}_S(w)} q_0^{\ell_0(w)} q^{\nu(w)} = (-q_0;q)_n [n]!_q \det[a_{ij}]_{i,j=-1,0,1,2,\ldots,n-1}$$

where

$$(x;q)_n := (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}), \qquad [n]!_q \quad := \frac{(q;q)_n}{(1-q)^n}$$

and

$$a_{ij} = \begin{cases} 0 & \text{for } j < i-1 \\ t_i - 1 & \text{for } j = i-1 \\ \frac{t_i}{(-q_0;q)_{j+1}[j+1]!_q} & \text{for } j \geq i = -1 \\ \frac{t_i}{[j-i+1]!_q} & \text{for } j \geq i \geq 0 \end{cases}$$

with the convention $t_{-1} = 1$. Thus the generating function in Corollary 3.3.3 is the evaluation of this determinant at $q_0 = 1, t_0 = 0$.

3.4. Palindromes revisited.

When s_0 is evenly-laced, the set of palindromes for (W^+, R) behaves much more like set of reflections in a Coxeter system (W, S), and plays a closely analogous role.

Definition 3.4.1. Given $w \in W^+$, define its set of *left-shortening palindromes* by

$$\mathtt{P}_L(w) := \{p \in \mathtt{P}(W^+) : \ell_{R \cup R^{-1}}(pw) < \ell_{R \cup R^{-1}}(w)\}.$$

Proposition 3.4.2. *Assume (W,S) has s_0 evenly-laced. Then for any $w \in W^+$, one has the following.*

(i) $\ell_{R\cup R^{-1}} = |\mathtt{P}_L(w)|$.

(ii) *(Strong exchange property) For any reduced $(R \cup R^{-1})^*$-word*

$$\mathbf{r} = (r^{(1)}, \ldots, r^{(\nu(w))})$$

factoring w, one has $\mathtt{P}_L(w) = \{p_k\}_{1 \leq k \leq \nu(w)}$ where

$$p_k := ((r^{(1)})^{-1}, (r^{(2)})^{-1}, \ldots, (r^{(k)})^{-1}, \ldots, (r^{(2)})^{-1}, (r^{(1)})^{-1}).$$

(iii) *The set $\mathtt{P}_L(w)$ determines w uniquely.*

3.5. Orders revisited.

Proposition 3.5.1. *When s_0 is evenly-laced, $u, w \in W^+$ satisfy $u \leq_{RW} w$ if and only if $\mathtt{P}_L(u) \subseteq \mathtt{P}_L(w)$.*

A similar statement holds for the left weak order $\leq_{LW}$, replacing left-shortening palindromes $\mathtt{P}_L(-)$ with right-shortening palindromes $\mathtt{P}_R(-)$.

Proposition 3.5.2. *When s_0 is evenly-laced, the left, right weak orders on W^+ are meet-semilattices.*

Proposition 3.5.3. *When s_0 is evenly-laced, $u, w \in W^+$ satisfy $u \leq_{LS} w$ if and only if for some (equivalently, every) reduced $(R \cup R^{-1})^*$-word $\mathbf{r} = (r^{(1)}, \ldots, r^{(\ell)})$ factoring w, there exists a reduced $(R \cup R^{-1})^*$-word factoring u which is a "subword" in the following sense:*

> *it can be obtained by deleting some of the $r^{(i)}$ from $\mathbf{r}$ and replacing any $r^{(i)}$ remaining that have an odd number of letters deleted to their right with their inverse $(r^{(i)})^{-1}$.*

A similar statement holds for the right strong order $\leq_{RS}$, replacing "right" with "left".

Proposition 3.5.4. *When s_0 is evenly-laced, the left, right strong orders on W^+ are thin*[3] *and shellable, and hence have every open interval homeomorphic to a sphere.*

Let w_0 be the unique longest element in a finite Coxeter group W.

---

[3] Recall that a graded poset is *thin* if every interval $[x, y]$ of rank 2 has exactly four elements, namely x, y and two others between them.

PROPOSITION 3.5.5. *When (W,S) has s_0 evenly-laced and W finite, $\tau(w_0)$ is the unique maximum element in all four (left or right, weak or strong) orders on W^+.*

REMARK 3.5.6. Note that when s_0 is not evenly-laced, the strong order need not be thin, as illustrated by the existence of several upper intervals of rank 2 having 5 elements in Figure 2.6. Also, the examples of $I_2(7), A_3$ show that one need not have a unique maximum element in any of these orders.

4. The case of a leaf node

The presentation (2) for W^+ becomes very close to a Coxeter presentation when s_0 is a *leaf* node, that is, s_0 commutes with $s_2, \ldots, s_n$, i.e., one has $m_{0i} = 2$ for $i = 2, \ldots, n$ (although m_{01} may be greater than 2).

PROPOSITION 4.0.7. *Let (W,S) be a Coxeter system with $S = \{s_0, s_1, \ldots, s_n\}$ and s_0 a leaf node. Then W^+ is generated by the set $R := \{r_i = s_0 s_i \mid s_i \in S \setminus s_0\}$ with the following presentation:*

$$W^+ \cong \langle R = \{r_1, \ldots, r_n\} : r_1^{m_{01}} = r_i^2 = (r_i r_j)^{m_{ij}} = e \textit{ for } 1 \leq i < j \leq n \rangle, \tag{4}$$

where m_{ij} is the order of $s_i s_j$ and s_1 is the neighbor of the leaf s_0.

DEFINITION 4.0.8. Call a presentation for an abstract group having the form in (4) a *nearly Coxeter* presentation, meaning that all but one of the generators r_i is an involution and all other relations are of the form $(r_i r_j)^{m_{ij}}$ for some $m_{ij} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$.

COROLLARY 4.0.9. *Every abstract group A with a nearly Coxeter presentation is isomorphic to the alternating subgroup W^+ for some Coxeter system (W,S).*

5. The case of an even leaf node

When the distinguished node s_0 is both a leaf and evenly-laced, that is, m_{01} is even and $m_{0j} = 2$ for $j = 2, 3, \ldots, n$, we shall say that s_0 is an *even leaf.* In this situation it will be shown that (W^+, R) has an amazingly close connection to the index 2 subgroup $W' := \ker \chi_0$ of W, which will turn out to have a Coxeter structure (W', S') of its own [4].

5.1. The Coxeter system (W', S').

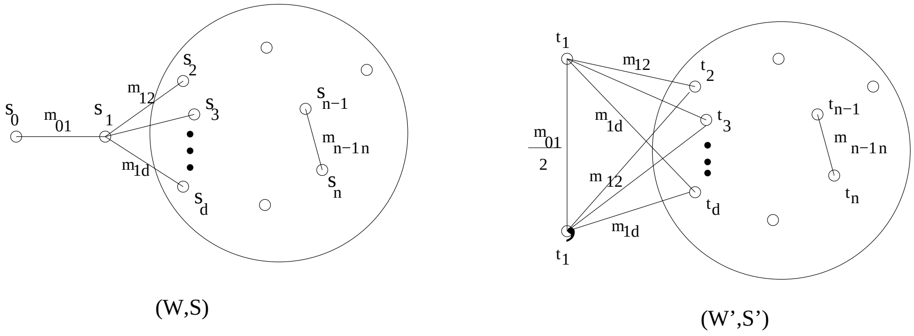
Since s_0 is also evenly laced, recall that one has the linear character $\chi_0 : W \to \{\pm 1\}$, taking value -1 on s_0 and $+1$ on all other $s_j \in S$. Let $W' := \ker \chi_0$, a subgroup of W of index 2.

Let $S' := \{t_1, t_2, \ldots, t_n\} \cup \{t_1'\}$ be a set, and consider the set map

$$\begin{array}{ccccl} S' & \xrightarrow{f} & W' & & \\ t_j & \overset{f}{\longmapsto} & s_j & & \text{for } j = 1, 2 \ldots, n. \\ t_1' & \overset{f}{\longmapsto} & s_0 s_1 s_0 & & \end{array}$$

---

[4] While the combinatorics of W^+ and W' seems to be similar, the combinatorics of other subgroups of index 2 seems to be different; in particular, no nearly Coxeter presentation for these groups is known; see, e.g., [**2**].



(W,S)

(W',S')

Figure 3. Schematic of the relation between the diagrams for a Coxeter system (W, S) with even leaf node s_0, and the Coxeter system (W', S') derived from it, closely connected to the alternating group W^+. The unique neighbor of s_0 has been labelled s_1, so that m_{01} is *even.*

Proposition 5.1.1. *The set map f above extends to an isomorphism*

$$
\begin{aligned}
W' \cong \langle S' = \{t_1, \ldots, t_n\} \cup \{t_1'\} :& \\
& (t_1')^2 = (t_i t_j)^{m_{ij}} = e \text{ for } 1 \le i \le j \le n, \\
& (t_1' t_j)^{m_{1j}} = e, \\
& (t_1' t_1)^{\frac{m_{01}}{2}} = e\rangle.
\end{aligned} \tag{5}
$$

which makes (W', S') a Coxeter system.

A schematic picture of the relation between the Coxeter diagrams of (W, S) and (W', S') is shown in Figure 5.1.

5.2. Relating W^+ to W'.

Proposition 5.2.1. *When s_0 is an even leaf in (W, S), the following map*

$$
\begin{array}{ccccc}
W^+ & \xrightarrow{\theta} & W' & & \\
w & \longmapsto & w \cdot s_0^{\ell_{R\cup R^{-1}}(w)} & = & \begin{cases} w & \text{if } w \in W' \\ ws_0 & \text{if } w \notin W' \end{cases}
\end{array}
$$

is a bijection, which is equivariant for the action of the subgroup $W^+ \cap W'$ by left-multiplication.

Note that the bijection $\theta : W^+ \to W'$ is *not* a group isomorphism, and that W^+, W' are not isomorphic as groups in general.

Proposition 5.2.2. *Let s_0 be an even leaf in (W, S). Then for any $w \in W^+$, the bijection θ has the following properties:*

(i) $\ell_{R\cup R^{-1}}(w) = \ell_{S'}(\theta(w))$.
(ii) *There is a bijection from the set of reduced $(R \cup R^{-1})^*$-words for w to the reduced $(S')^*$-words for $\theta(w)$*

(iii) *There is a bijection from* $\mathrm{Nasc}_{R\cup R^{-1}}(w)$ *to* $\mathrm{Des}_{S'}(\theta(w))$.
(iv) *The map* θ *is a poset isomorphism* $(W^+, \leq_{RW}) \to (W', \leq_{RW})$.
(v) *The map* θ *is a poset isomorphism* $(W^+, \leq_{RS}) \to (W', \leq_S)$, *where* $(W', \leq_S)$ *denotes usual strong Bruhat order on* (W', S')

COROLLARY 5.2.3. *When* s_0 *is an even leaf in* (W, S), *one has*

$$\sum_{w\in W^+} t^{\mathrm{nasc}_{R\cup R^{-1}}(w)} q^{\ell_{R\cup R^{-1}}(w)} = \sum_{w\in W'} t^{\mathrm{des}_{S'}(w)} q^{\ell_{S'}(w)}, \tag{6}$$

where $\mathrm{nasc}_{R\cup R^{-1}}(w) := |\mathrm{Nasc}_{R\cup R^{-1}}(w)|$.

EXAMPLE 5.2.4. For (W, S) of affine type $\tilde{C_n}$, one has (W', S') equal to the affine type $\tilde{B}_n$, and

$$W^+(R\cup R^{-1}; q) = W'(S'; q) = \frac{[2]_q}{1-q^1}\frac{[4]_q}{1-q^3}\frac{[6]_q}{1-q^5}\cdots\frac{[2n]_q}{1-q^{2n-1}}.$$

6. Acknowledgements

The authors thank John Stembridge for pointing them to the Bourbaki exercise from Section 2.1 at an early stage of this investigation.

References

[1] E. Babson and V. Reiner, Coxeter-like complexes. *Discrete Math. Theor. Comput. Sci.* **6** (2004), 223–251 (electronic).
[2] D. Bernstein, A maj-inv bijection for $C_2 \wr A_n$. preprint 2005, arXiv.math.CO/0509239.
[3] A. Björner, Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings. *Adv. in Math.* **52** (1984), 173–212.
[4] A. Björner and F. Brenti, Combinatorics of Coxeter groups. Graduate Texts in Mathematics, 231. Springer, New York, 2005.
[5] N. Bourbaki, Lie Groups and Lie Algebras. English translation by Andrew Pressley, Springer 2002.
[6] F. Brenti, V. Reiner and Y. Roichman, Alternating subgroups of Coxeter groups. preprint 2006.
[7] V.V. Deodhar, On the root system of a Coxeter group. *Comm. Algebra* **10** (1982), 611–630.
[8] M.J. Dyer, Reflection subgroups of Coxeter systems. *J. Algebra* **135** (1990), 57–73.
[9] J.E. Humphreys, Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics. **29**. Cambridge University Press, Cambridge, 1990.
[10] I.G. Macdonald, The Poincaré series of a Coxeter group. *Math. Ann.* **199** (1972), 161–174.
[11] H. Mitsuhashi, The q-analogue of the alternating group and its representations. *J. Algebra* **240** (2001), 535–558.
[12] A. Regev and Y. Roichman, Permutation statistics on the alternating group. *Adv. in Appl. Math.* **33** (2004), 676–709.
[13] V. Reiner, The distribution of descents and length in a Coxeter group. *Electron. J. Combin.* **2** (1995), R25 (electronic).
[14] R.P. Stanley, Some aspects of groups acting on finite posets. *J. Combin. Theory Ser. A* **32** (1982), 132–161.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÁ DI ROMA "TOR VERGATA", VIA DELLA RICERCA SCIENTIFICA, 00133 ROMA, ITALY
E-mail address: `brenti@mat.uniroma2.it`

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455, USA
E-mail address: `reiner@math.umn.edu`

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, 52900 RAMAT-GAN, ISRAEL
E-mail address: `yuvalr@math.biu.ac.il`