

UNIFICATION OF THE QUINTUPLE AND SEPTUPLE PRODUCT IDENTITIES

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ABSTRACT. By combining the functional equation method with Jacobi's triple product identity, we establish a general equation with five free parameters on the modified Jacobi theta function, which can be considered as the common generalization of the quintuple, sextuple and septuple product identities. Several known theta function formulae and new identities are proved consequently.

1. INTRODUCTION AND NOTATION

For two indeterminate q and x with $|q| < 1$, the q -shifted factorial of infinite order reads as

$$(x; q)_{\infty} = \prod_{i=0}^{\infty} (1 - xq^i) = (1 - x)(1 - qx)(1 - qx^2) \cdots .$$

Then we define the modified Jacobi theta function by

$$\langle x; q \rangle_{\infty} = (q; q)_{\infty} (x; q)_{\infty} (q/x; q)_{\infty} .$$

Their product forms are abbreviated respectively as

$$\begin{aligned} (\alpha, \beta, \cdots, \gamma; q)_{\infty} &= (\alpha; q)_{\infty} (\beta; q)_{\infty} \cdots (\gamma; q)_{\infty}, \\ \langle \alpha, \beta, \cdots, \gamma; q \rangle_{\infty} &= \langle \alpha; q \rangle_{\infty} \langle \beta; q \rangle_{\infty} \cdots \langle \gamma; q \rangle_{\infty}. \end{aligned}$$

There are several important theta function identities in mathematical literature. Perhaps the simplest and the most significant one is Jacobi's triple product identity [23] (see [18, §1.6] also):

$$\langle x; q \rangle_{\infty} = \sum_{n=-\infty}^{+\infty} (-1)^n q^{\binom{n}{2}} x^n .$$

For different proofs and applications, we refer the reader to the papers by Andrews [2], Chu [8], Ewell [10], Lewis [25], Mordell [27] and Wright [32]. The next one is the celebrated quintuple product identity attributed originally to Watson [31]. However, it can also be found in Ramanujan's lost notebook, which has been cleared by Berndt [4, P 83]. Later in 1995, Ewell [11] found a sextuple product identity, which has both beautiful form and interesting applications to congruences of

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partition function (see Ewell [12, 13, 14]). Hirschhorn [21] found, first in 1983, the septuple product identity, which was subsequently rediscovered by Farkas and Kra [15] in 1999. More recently, Chapman [6], Foata-Han [16], Garvan [17] and Kongsirwong-Liu [24] provided different proofs. The informed reader may notice that some identities just mentioned can be recovered from the identities for affine root systems due to Macdonald [26] such as the quintuple product identity (from BC_1) and the septuple product identity (from BC_2), as observed by Rosengren and Schlosser [28, §6].

The purpose of the present paper is to unify all the identities just mentioned into a single formula. By means of the functional equation method and Jacobi's triple product identity, we shall show a general equation on the modified Jacobi theta function which covers several known and new theta function identities. The main theorem and its proof will be given in the next section. As applications, we shall systematically review the known identities such as the quintuple, sextuple and septuple products and establish several new theta function identities through Section 3 to 5 in the rest of the paper.

2. MAIN THEOREM AND PROOF

By combining the functional equations with Jacobi's triple product identity, we prove the following fundamental result.

Theorem 1. *Let α , β and γ be three natural integers with $\gcd(\alpha, \gamma) = 1$ and $\lambda = 1 + \alpha\beta^2\gamma$. For two indeterminate x and y with $x \neq 0$ and $y \neq 0$, there holds the algebraic identity:*

$$\begin{aligned} \langle x; q^\alpha \rangle_\infty \langle x^{\beta\gamma} y; q^\gamma \rangle_\infty &= \sum_{\ell=0}^{\alpha\beta^2\gamma} (-1)^\ell q^{\binom{\ell}{2}\alpha} x^\ell \left\langle (-1)^{\alpha\beta} x^\lambda y^{\alpha\beta} q^{\binom{\alpha\beta}{2}\gamma + \ell\alpha}; q^{\lambda\alpha} \right\rangle_\infty \\ &\quad \times \left\langle (-1)^{\beta\gamma} y q^{\binom{\beta\gamma+1}{2}\alpha - \ell\alpha\beta\gamma}; q^{\lambda\gamma} \right\rangle_\infty. \end{aligned}$$

Proof. For the bivariate function $f(x, y)$ defined by the infinite product

$$f(x, y) = \langle x; q^\alpha \rangle_\infty \langle x^{\beta\gamma} y; q^\gamma \rangle_\infty$$

it is easy to see that $f(x, y)$ is analytic within $0 < |x| < \infty$. Then we can expand it as a Laurent series in x :

$$f(x, y) = \sum_{k=-\infty}^{+\infty} \Omega_k(y) x^k.$$

From the definition of $f(x, y)$, it is trivial to check the functional equation

$$f(x, y) = (-1)^{1+\alpha\beta} q^{\binom{\alpha\beta}{2}\gamma} x^\lambda y^{\alpha\beta} f(q^\alpha x, y),$$

which corresponds to the recurrence relation

$$\Omega_k(y) = (-1)^{1+\alpha\beta} y^{\alpha\beta} q^{(k-\lambda)\alpha + \binom{\alpha\beta}{2}\gamma} \Omega_{k-\lambda}(y).$$

Iterating this relation for k -times, we find that there exist $\lambda = 1 + \alpha\beta^2\gamma$ formal power series $\Omega_\ell(y)$ with $0 \leq \ell \leq \alpha\beta^2\gamma$ such that there hold

$$\Omega_{k\lambda+\ell}(y) = (-1)^{k+k\alpha\beta} y^{k\alpha\beta} q^{\lambda\binom{k}{2}\alpha+k\binom{\alpha\beta}{2}\gamma+k\ell\alpha} \Omega_\ell(y) \quad \text{where } k \in \mathbb{Z}.$$

By invoking Jacobi's triple product identity, we can also expand $f(x, y)$ directly into the following double series:

$$f(x, y) = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} (-1)^{i+j} q^{\binom{i}{2}\alpha+\binom{j}{2}\gamma} x^{i+j\beta\gamma} y^j.$$

Equating the coefficients of x^ℓ in the two formal power series expansions of $f(x, y)$, we find that

$$\begin{aligned} \Omega_\ell(y) &= \sum_{j=-\infty}^{+\infty} (-1)^{j+\ell+j\beta\gamma} y^j q^{\binom{\ell-j\beta\gamma}{2}\alpha+\binom{j}{2}\gamma} \\ &= (-1)^\ell q^{\binom{\ell}{2}\alpha} \sum_{j=-\infty}^{+\infty} (-1)^{j+j\beta\gamma} y^j q^{\lambda\binom{j}{2}\gamma+j\binom{\beta\gamma+1}{2}\alpha-j\ell\alpha\beta\gamma}. \end{aligned}$$

Therefore, $f(x, y)$ can be reformulated as

$$\begin{aligned} f(x, y) &= \sum_{\ell=0}^{\alpha\beta^2\gamma} \sum_{k=-\infty}^{+\infty} \Omega_{k\lambda+\ell}(y) x^{k\lambda+\ell} = \sum_{\ell=0}^{\alpha\beta^2\gamma} (-1)^\ell q^{\binom{\ell}{2}\alpha} x^\ell \\ &\times \sum_{k=-\infty}^{+\infty} (-1)^{k+k\alpha\beta} x^{k\lambda} y^{k\alpha\beta} q^{\lambda\binom{k}{2}\alpha+k\binom{\alpha\beta}{2}\gamma+k\ell\alpha} \\ &\times \sum_{j=-\infty}^{+\infty} (-1)^{j+j\beta\gamma} y^j q^{\lambda\binom{j}{2}\gamma+j\binom{\beta\gamma+1}{2}\alpha-j\ell\alpha\beta\gamma}. \end{aligned}$$

Applying twice Jacobi's triple product identity to the double sum just displayed, we establish the equation stated in the theorem. \square

The simplest case $\alpha = \beta = \gamma = 1$ of Theorem 1 reads as the following sextuple product identity, essentially discovered by Ewell [11, 12, 14].

Corollary 2 (The sextuple product identity).

$$\langle x, xy; q \rangle_\infty = \langle -qy, -x^2y; q^2 \rangle_\infty - x \langle -y, -qx^2y; q^2 \rangle_\infty.$$

In fact, making the parameter replacements $q \rightarrow q^2$, $x \rightarrow -qxy$ and $y \rightarrow 1/y^2$ and then applying Jacobi's triple product identity, we see that the last equation is equivalent to the sextuple product identity due to Ewell [12, Eq 2.2]:

$$\begin{aligned} \langle -qxy, -qx/y; q^2 \rangle_\infty &= \sum_{i=-\infty}^{+\infty} q^{2i^2} x^{2i} \sum_{j=-\infty}^{+\infty} q^{2j^2} y^{2j} \\ &+ q \sum_{i=-\infty}^{+\infty} q^{2i(i+1)} x^{2i+1} \sum_{j=-\infty}^{+\infty} q^{2j(j+1)} y^{2j+1}. \end{aligned}$$

If we let $q \rightarrow q^2$, $x \rightarrow -q$ and $y \rightarrow 1$ in Corollary 2, then we can get immediately another identity due to Ewell [13, Thm 2.1]:

$$\prod_{n=1}^{\infty} (1 - q^{4n-2})^2 (1 + q^{2n-1})^4 = \prod_{n=1}^{\infty} (1 + q^{4n-2})^4 + 4q \prod_{n=1}^{\infty} (1 + q^{4n})^4.$$

For more applications of the sextuple product identity, the interested reader may consult Ewell [11, 12, 13, 14].

3. THE QUINTUPLE PRODUCT IDENTITIES

This section reviews the application of Theorem 1 to quintuple product identities. We first state the case $\beta = 1$ of Theorem 1 as follows.

Proposition 3. *Let α and γ be two natural integers with $\gcd(\alpha, \gamma) = 1$. For two indeterminate x and y with $x \neq 0$ and $y \neq 0$, there holds the algebraic identity:*

$$\begin{aligned} \langle x; q^\alpha \rangle_\infty \langle x^\gamma y; q^\gamma \rangle_\infty &= \sum_{\ell=0}^{\alpha\gamma} (-1)^\ell q^{\binom{\ell}{2}\alpha} x^\ell \left\langle (-1)^\alpha x^{1+\alpha\gamma} y^\alpha q^{\binom{\alpha}{2}\gamma+\ell\alpha}; q^{\alpha(1+\alpha\gamma)} \right\rangle_\infty \\ &\quad \times \left\langle (-1)^\gamma y q^{\binom{\gamma+1}{2}\alpha-\ell\alpha\gamma}; q^{\gamma(1+\alpha\gamma)} \right\rangle_\infty. \end{aligned}$$

When $\alpha = 2$ and $\gamma = 1$, we find from Proposition 3 the following generalized quintuple product identity.

Corollary 4.

$$\begin{aligned} \langle x; q \rangle_\infty \langle xy; q^2 \rangle_\infty &= \langle -qy; q^3 \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{3i^2-2i} x^{3i} y^i \\ &\quad - \langle -q^3 y; q^3 \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{3i^2} x^{3i+1} y^{i+1} \\ &\quad + \langle -q^2 y; q^3 \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{3i^2+2i} x^{3i+2} y^{i+1}. \end{aligned}$$

Instead, when $\alpha = 1$ and $\gamma = 2$, we recover from Proposition 3 another generalized quintuple product identity.

Corollary 5 (Hirschhorn [22, Eq 2]).

$$\begin{aligned} \langle x; q \rangle_\infty \langle x^2 y; q^2 \rangle_\infty &= \langle q^3 y; q^6 \rangle_\infty \sum_{i=-\infty}^{+\infty} q^{3\binom{i}{2}} x^{3i} y^i \\ &\quad - \langle qy; q^6 \rangle_\infty \sum_{i=-\infty}^{+\infty} q^{3\binom{i}{2}+i} x^{3i+1} y^i \\ &\quad - \langle q^5 y; q^6 \rangle_\infty \sum_{i=-\infty}^{+\infty} q^{3\binom{i}{2}+2i} x^{3i+2} y^{i+1}. \end{aligned}$$

We remark that under the parameter replacement $y \rightarrow xy$, the left member displayed in Corollary 4 becomes that in Corollary 5. However the corresponding right members are not euqivalent.

For $x \rightarrow -qx$ and $y \rightarrow -q^{-1}$, Corollary 5 reduces to the following identity due to Stanton [29, 1986], who derived it through sign variations of the Macdonald identities.

Example 6 (Stanton [29]).

$$\begin{aligned} \langle -qx; q \rangle_{\infty} \langle -qx^2; q^2 \rangle_{\infty} &= \langle -1; q^6 \rangle_{\infty} \sum_{i=-\infty}^{+\infty} q^{3\binom{i}{2}+3i+1} x^{3i+1} \\ &+ \langle -q^2; q^6 \rangle_{\infty} \sum_{i=-\infty}^{+\infty} q^{3\binom{i}{2}+2i} \{1 + q^{2i+1}x^2\} x^{3i}. \end{aligned}$$

Letting $y = 1$ and $y = q$ in Corollary 5, we further get respectively the following two quintuple product identities.

Example 7.

$$\begin{aligned} \langle x; q \rangle_{\infty} \langle x^2; q^2 \rangle_{\infty} &= \langle q^3; q^6 \rangle_{\infty} \sum_{i=-\infty}^{+\infty} q^{3\binom{i}{2}} x^{3i} \\ &- \langle q; q^6 \rangle_{\infty} \sum_{i=-\infty}^{+\infty} q^{3\binom{i}{2}+i} (1 + q^i x) x^{3i+1}. \end{aligned}$$

Example 8 (The quintuple product identity).

$$\langle x; q \rangle_{\infty} \langle qx^2; q^2 \rangle_{\infty} = (q^2; q^2)_{\infty} \sum_{i=-\infty}^{+\infty} q^{3\binom{i}{2}+i} (1 - xq^i) x^{3i}.$$

The last one is the so-called quintuple product identity. For the historical note, different proofs and extensions about this celebrated identity, the reader can refer to [1, 3, 5, 9, 19, 22, 30]. There is also an algebraic interpretation in terms of Macdonald's identity for non-exceptional root system BC_1 (cf. [28, Prop 6.1]. In particular, two finite forms of the quintuple product identity have been given in the recent papers [7] and [20].

4. THE SEPTUPLE PRODUCT IDENTITIES

In order to derive the generalized septuple product identities, we state first the case $\beta = 2$ of Theorem 1 as follows.

Proposition 9. *Let α and γ be two natural integers with $\gcd(\alpha, \gamma) = 1$. For two indeterminate x and y with $x \neq 0$ and $y \neq 0$, there holds the algebraic identity:*

$$\begin{aligned} \langle x; q^\alpha \rangle_\infty \langle x^{2\gamma} y; q^\gamma \rangle_\infty &= \sum_{\ell=0}^{4\alpha\gamma} (-1)^\ell q^{\binom{\ell}{2}\alpha} x^\ell \left\langle x^{1+4\alpha\gamma} y^{2\alpha} q^{\binom{2\alpha}{2}\gamma+\ell\alpha}; q^{\alpha(1+4\alpha\gamma)} \right\rangle_\infty \\ &\quad \times \left\langle y q^{\binom{2\gamma+1}{2}\alpha-2\ell\alpha\gamma}; q^{\gamma(1+4\alpha\gamma)} \right\rangle_\infty. \end{aligned}$$

When $\alpha = \gamma = 1$, replacing q by q^2 in Proposition 9 leads to the following generalized septuple product identity.

Corollary 10.

$$\begin{aligned} \langle x, x^2 y; q^2 \rangle_\infty &= \langle q^6 y; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2-3i} x^{5i} y^{2i} \\ &\quad - \langle q^2 y; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2-i} x^{5i+1} y^{2i} \\ &\quad - \langle q^8 y; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2+i} x^{5i+2} y^{2i+1} \\ &\quad + \langle q^4 y; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2+3i} x^{5i+3} y^{2i+1} \\ &\quad - \langle y; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2+5i+2} x^{5i+4} y^{2i+1}. \end{aligned}$$

The two particular cases of interest may be displayed as follows.

Example 11 (The septuple product identity: $x \rightarrow x$ and $y \rightarrow 1$ in Corollary 10).

$$\begin{aligned} \langle x, x^2; q^2 \rangle_\infty &= \langle q^4; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2-3i} \{1 + q^{6i} x^3\} x^{5i} \\ &\quad - \langle q^2; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2-i} \{1 + q^{2i} x\} x^{5i+1}. \end{aligned}$$

We point out that this septuple product identity first appeared in Hirschhorn [21] explicitly, even though it has been attributed erroneously to Farkas-Kra [15] by Foata-Han [16, Eq 1.6] and Chapman [6, Thm 1]. The algebraic formulation in terms of Macdonald's identity for non-exceptional root system BC_2 (with $x_2 = -1$) can be found in [28, Prop 6.1].

Example 12 (Kongsiriwong-Liu [24, Eq 7.36]: $x \rightarrow qx$ and $y \rightarrow q^{-1}$ in Corollary 10).

$$\begin{aligned} \langle qx, qx^2; q^2 \rangle_\infty &= \langle q^5; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2} x^{5i} \\ &\quad - \langle q; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2+2i+1} \{1 - q^{6i+3} x^3\} x^{5i+1} \\ &\quad - \langle q^3; q^{10} \rangle_\infty \sum_{i=-\infty}^{+\infty} (-1)^i q^{5i^2+4i+1} \{1 - q^{2i+1} x\} x^{5i+2}. \end{aligned}$$

5. FURTHER THETA FUNCTION IDENTITIES

For the presence of five free parameters, we may further specialize Theorem 1 to numerous theta function identities. Leaving the parameter γ invariant, this section collects four theta function identities, just for examples, which have appeared previously in Kongsiriwong-Liu [24]. Here all the parameter settings refer to Theorem 1 under the same base substitution $q \rightarrow q^2$.

Example 13 ([24, Thm 6]: $\alpha = 1$ and $\beta = 1$: $x \rightarrow qx$ and $y \rightarrow q^{-\gamma}$).

$$\begin{aligned} \langle qx; q^2 \rangle_\infty \langle x^\gamma; q^{2\gamma} \rangle_\infty &= \sum_{\ell=0}^{\gamma} (-1)^\ell q^{\ell^2} \left\{ \sum_{i=-\infty}^{+\infty} (-1)^{(\gamma+1)i} q^{(\gamma^2+\gamma)i^2 - (2\ell+1)\gamma i} \right\} \\ &\quad \times \left\{ \sum_{j=-\infty}^{+\infty} q^{(\gamma+1)j^2 + (2\ell-\gamma)j} x^{(\gamma+1)j+\ell} \right\}. \end{aligned}$$

Example 14 ([24, Thm 7]: $\alpha = 1$ and $\beta = 1$: $x \rightarrow qx$ and $y \rightarrow 1$).

$$\begin{aligned} \langle qx; q^2 \rangle_\infty \langle q^\gamma x^\gamma; q^{2\gamma} \rangle_\infty &= \sum_{\ell=0}^{\gamma} (-1)^\ell q^{\ell^2} \left\{ \sum_{i=-\infty}^{+\infty} (-1)^{(\gamma+1)i} q^{(\gamma^2+\gamma)i^2 - 2\ell\gamma i} \right\} \\ &\quad \times \left\{ \sum_{j=-\infty}^{+\infty} q^{(\gamma+1)j^2 + 2\ell j} x^{(\gamma+1)j+\ell} \right\}. \end{aligned}$$

Example 15 ([24, Thm 8]: $\alpha = 1$ and $\beta = 2$: $x \rightarrow x$ and $y \rightarrow 1$).

$$\begin{aligned} \langle x; q^2 \rangle_\infty \langle x^{2\gamma}; q^{2\gamma} \rangle_\infty &= \sum_{\ell=0}^{4\gamma} (-1)^\ell q^{\ell^2 - \ell} \left\{ \sum_{i=-\infty}^{+\infty} (-1)^i q^{(4\gamma^2+\gamma)i^2 + (1-4\ell)\gamma i} \right\} \\ &\quad \times \left\{ \sum_{j=-\infty}^{+\infty} (-1)^j q^{(4\gamma+1)j^2 + (2\ell-2\gamma-1)j} x^{(4\gamma+1)j+\ell} \right\}. \end{aligned}$$

Example 16 ([24, Thm 9]: $\alpha = 1$ and $\beta = 2$: $x \rightarrow qx$ and $y \rightarrow q^{-\gamma}$).

$$\begin{aligned} \langle qx; q^2 \rangle_{\infty} \langle q^{\gamma} x^{2\gamma}; q^{2\gamma} \rangle_{\infty} &= \sum_{\ell=0}^{4\gamma} (-1)^{\ell} q^{\ell^2} \left\{ \sum_{i=-\infty}^{+\infty} (-1)^i q^{(4\gamma^2+\gamma)i^2-4\ell\gamma i} \right\} \\ &\quad \times \left\{ \sum_{j=-\infty}^{+\infty} (-1)^j q^{(4\gamma+1)j^2+2\ell j} x^{(4\gamma+1)j+\ell} \right\}. \end{aligned}$$

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