

Involutions avoiding the class of permutations in \mathfrak{S}_k with prefix 12

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ABSTRACT. An involution π is said to be τ -avoiding if it does not contain any subsequence having all the same pairwise comparisons as τ . This paper concerns the enumeration of involutions which avoid a set \mathcal{A}_k of subsequences increasing both in number and in length at the same time. Let \mathcal{A}_k be the set of all the permutations $12\pi_3 \dots \pi_k$ of length k . For $k = 3$ the only subsequence in \mathcal{A}_k is 123 and the 123-avoiding involutions of length n are enumerated by the central binomial coefficients $\binom{n}{\lfloor n/2 \rfloor}$. For $k = 4$ we give a combinatorial explanation that shows the number of involutions of length n avoiding \mathcal{A}_4 is the same as the number of symmetric Schröder paths of length $n - 1$. For each $k \geq 3$ we determine the generating function for the number of involutions avoiding the subsequences in \mathcal{A}_k , according to length, first entry and number of fixed points.

1. Introduction

Let $[d]$ denote a totally ordered alphabet on d letters, and let \mathfrak{S}_n denote the set of permutations of $[n] = \{1, \dots, n\}$, written in one-line notation, and suppose $\pi \in \mathfrak{S}_n$. We write $|\pi|$ to denote the length of π , namely n , and for all $i \in [n]$, we write π_i to denote the i -th element of π . Let $\alpha = \alpha_1 \dots \alpha_m \in [p_1]^m$, $\beta = \beta_1 \dots \beta_m \in [p_2]^m$. We say that α is *order-isomorphic* to β if for all $1 \leq i < j \leq m$ one has $\alpha_i < \alpha_j$ if and only if $\beta_i < \beta_j$. For two permutations $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$, an *occurrence* of τ in π is a subsequence $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that $(\pi_{i_1}, \dots, \pi_{i_k})$ is order-isomorphic to τ ; in such a context τ is usually called the *pattern*. We say that π *avoids* τ , or is *τ -avoiding*, if there is no occurrence of τ in π . A natural generalization of single pattern avoidance is *subset avoidance*, that is, we say that $\pi \in \mathfrak{S}_n$ avoids a subset $T \subseteq \mathfrak{S}_k$ if π avoids all $\tau \in T$. The set of all τ -avoiding (resp. T -avoiding) permutations of length n is denoted $\mathfrak{S}_n(\tau)$ (resp. $\mathfrak{S}_n(T)$).

Several authors have considered the case of general k in which T enjoys various algebraic properties. Barucci et al. [2] treat the case of permutations avoiding the collection of permutations in \mathfrak{S}_k that have suffix $(k-1)k$. Adin and Roichman [1] look at the case where T is a Kazhdan–Lusztig cell of \mathfrak{S}_k , or, equivalently, a Knuth equivalence class (see [12, Vol. 2, Ch. A1]). Mansour and Vainshtein [10] consider the situation where T is a maximal parabolic subgroup of \mathfrak{S}_k . In the current paper an analogous result is established for pattern-avoiding involutions. Simion and Schmidt [11] considered

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the first cases of pattern-avoiding involutions, which was continued in Gouyou-Beauchamps [5] and Gessel [6] for increasing patterns, and subsequently in Guibert's Ph.D. thesis [7].

Kremer [8, Corollary 9] has shown that $|\mathfrak{S}_n(1243, 2143)|$, the number of permutations in \mathfrak{S}_n that avoid both 1243 and 2143, is given by r_{n-1} , for all $n \geq 0$, where r_n is the n -th Schröder number, defined by $r_0 = 1$ and $r_n = r_{n-1} + \sum_{i=1}^n r_{i-1}r_{n-i}$ for $n \geq 1$. As a result, for all $n \geq 0$, the set $\mathfrak{S}_{n+1}(1243, 2143)$ is in bijection with the set of *Schröder paths* of length n . These are the lattice paths from $(0, 0)$ to (n, n) which contain only east $E = (1, 0)$, north $N = (0, 1)$, and diagonal $D = (1, 1)$ steps and which do not pass below the line $y = x$. A *symmetric Schröder path* of length n is a Schröder path of length n which is symmetric about the line $x + y = n$.

We say π is an *involution* whenever $\pi_{\pi_i} = i$ for all $i \in [n]$. We denote by \mathcal{I}_n and I_n the set of involutions in S_n and its cardinality, respectively. We say that i is a fixed point of a permutation π if $\pi_i = i$. Define $J_n(p)$ to be the polynomial $\sum_{j=0}^n I_{n;j} p^j$, where $I_{n;j}$ is the number of involutions in \mathcal{I}_n with j fixed points. For example, $J_0(p) = 1$, $J_1(p) = p$, $J_2(p) = 1 + p^2$, and $J_3(p) = 3p + p^3$. It is not hard to see that the polynomial $J_n(p)$ satisfies the recurrence relation $J_n(p) = pJ_{n-1}(p) + (n-1)J_{n-2}(p)$, $n \geq 1$, with the initial condition $J_0(p) = 1$. The exponential generating function for the sequence $J_n(p)$ is given by $e^{px+x^2/2}$.

The main result of this paper can be formulated as follows. Let \mathcal{A}_k be the class of permutations in \mathfrak{S}_k with prefix 12, that is,

$$\mathcal{A}_k = \{\pi_1\pi_2 \dots \pi_k \in \mathfrak{S}_k \mid \pi_1 = 1, \pi_2 = 2\}.$$

THEOREM 1.1. *Let $k \geq 2$. Then the generating function for the number of \mathcal{A}_k -avoiding involutions of length n with m fixed points is given by*

$$\begin{aligned} \sum_{n \geq 0} \sum_{\pi \in \mathcal{I}_n} x^n p^{\#\text{fixed points in } \pi} &= \sum_{j=0}^{k-3} J_j(p) x^j - \frac{x^{k-3}}{2} J_{k-2}(p) \left(p + (p(k-3)x^2 - 2x - p)u_0(x) \right) \\ &\quad - \frac{x^{k-4}}{2} J_{k-3}(p) \left(x + p - (x^3(k-3) - px^2(k-1) + x + p)u_0(x) \right), \end{aligned}$$

where $u_0(x) = 1/\sqrt{1 - 2(k-1)x^2 + (k-3)^2x^4}$.

The proof is given in Section 3. Theorem 1.1 with $k = 3$ and $p = 1$ shows the generating function for the number 123-avoiding involutions of length n to be $\frac{2x}{2x-1+\sqrt{1-4x^2}} = \sum_{n \geq 0} \binom{n}{\lfloor n/2 \rfloor} x^n$ (see [11]). Moreover, Theorem 1.1 with $k = 3$ and $p = 0$ gives the number of 123-avoiding involutions of length $2n$ without fixed points to be $\frac{1}{2} \binom{2n}{\lfloor n/2 \rfloor}$. Also, Theorem 1.1 with $k = 4$ and $p = 1$ gives the generating function for the number {1234, 1243}-avoiding involutions of length n to be $\frac{1-x}{2} + \frac{1+x}{2} \sqrt{\frac{1+2x-x^2}{1-2x-x^2}}$. In Section 2 we give a combinatorial explanation for this result and prove that the number {1234, 1243}-avoiding involutions of length n is the same as the number of symmetric Schröder paths of length $n-1$.

2. Symmetric Schröder paths and {1234, 1243}-avoiding involutions

In this section we give a combinatorial explanation for why the number of *symmetric Schröder paths* of length n (for definition, see below) is the same as the number of {1234, 1243}-avoiding involutions on $n+1$ elements. This explanation is via the existence of a bijection between the two sets. The bijection is in fact the composition of three bijections which are explained in the following three subsections. Unfortunately the composition does not appear to reduce to one that has a simple description.

2.1. Schröder paths and $\{1243, 2143\}$ -avoiding involutions. The involutions in $\mathcal{I}_n(1243, 2143)$ may be related to Schröder paths via the bijection φ in Egge and Mansour [4], which we now describe.

Let S_n be the set of all Schröder paths from $(0, 0)$ to (n, n) and SS_n the subcollection which are symmetric about the line $x+y = n$. For such a path $p \in S_n$ let us select a sequence of diagonals (parallel to $y = x$ and passing through the points $(0.5, 0.75)$ of the unit squares with integer coordinates) that are contained within (and on) the bounded region between p and the line $y = x$ in the following manner;

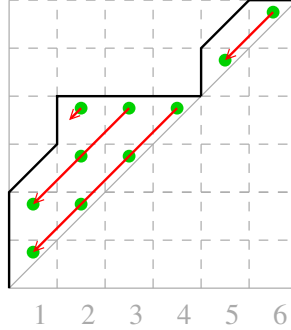
Let p be a Schröder path p from $(0, 0)$ to (n, n) and s_i be the transposition $(i, i + 1)$.

Step 1: If a square with integer coordinates $(i - 1, m - 1)$, $(i, m - 1)$, $(i - 1, m)$ and (i, m) has interior point $(i - 0.5, m - 0.25)$ within the region between the path and the line $y = x$ then earmark this square and label it s_i . Let $j = 1$.

Step 2: Choose the rightmost earmarked square that is not yet marked (with label s_k , say) and suppose one may see (to the south-west) as far as a square with label s_l , where $l \leq k$ of course. Mark both these squares and all those in-between. Let $\sigma_j = s_k s_{k-1} \dots s_l$. If there are no further earmarked squares then go to step 3. Otherwise increase j by 1 and repeat.

Step 3: Let $\varphi(p) = \sigma_j \dots \sigma_2 \sigma_1(n + 1, n, \dots, 1)$.

EXAMPLE 2.1. Consider the path $p \in S_6$ in the diagram.



The green points show those squares that have been earmarked. We have $\sigma_1 = s_6 s_5$, $\sigma_2 = s_4 s_3 s_2 s_1$, $\sigma_3 = s_3 s_2 s_1$ and $\sigma_4 = s_2$. So

$$\begin{aligned} \varphi(p) &= \sigma_4 \sigma_3 \sigma_2 \sigma_1(7, 6, 5, 4, 3, 2, 1) \\ &= s_2 s_3 s_2 s_1 s_4 s_3 s_2 s_1 s_6 s_5(7, 6, 5, 4, 3, 2, 1) \\ &= (5, 2, 4, 6, 7, 1, 3). \end{aligned}$$

We generalise our notation slightly. If $\pi \in \mathfrak{S}_n$ and $A = [a_1, a_2, \dots, a_n]$ is an increasing sequence of numbers, then let $\pi \leftarrow A$ be the sequence $(b_{a_1}, \dots, b_{a_n})$ in which $b_{a_i} = a_{\pi_i}$ for all $1 \leq i \leq n$ and all other positions are left empty. Note that $(\pi_1, \dots, \pi_n) \leftarrow [1, \dots, n] = \pi$. We use \cup to denote the ‘filling up’ of the (possibly) empty spaces in such a sequence by transpositions. (The reason for this is that we wish to end up with a sequence which will be a permutation, and the \leftarrow operation does not necessarily guarantee a permutation.)

EXAMPLE 2.2. We have $(1, 4, 2, 3, 5, 6) \leftarrow [1, 2, 4, 5, 6, 7] = (1, 5, \quad, 2, 4, 6, 7)$ and so $((1, 4, 2, 3, 5, 6) \leftarrow [1, 2, 4, 5, 6, 7]) \cup (3, 8) = (1, 5, 8, 2, 4, 6, 7, 3)$.

PROPOSITION 2.3.

(i) For all $n \geq 2$ and $p = Dp'D \in SS_n$,

$$\varphi(Dp'D) = (\varphi(p') \leftarrow [2, \dots, n]) \cup (1, n+1).$$

(ii) For all $n \geq 1$ and $p = Np'E \in SS_n$ where $p' \in SS_n$, $\varphi(p) = (\varphi(p'), n+1)$.

(iii) For all $n \geq 0$ and $p = NqEp'Nq^\perp E \in SS_n$, where $q \in S_{m-2}$, $p \in SS_{n-2m}$ and $Nq^\perp E$ is the reflection of NqE from $(n-m, n-m)$ to (n, n) , we have

$$\begin{aligned} \varphi(p) &= \sigma(qp'q^\perp)\{1, \dots, m, m+2, \dots, n\} \cup (n+1, m+1) \\ &= (\varphi(qp'q^\perp) \leftarrow [1, \dots, m, m+2, \dots, n]) \cup (n+1, m+1). \end{aligned}$$

PROOF. (i) If the path begins and ends with D , then s_1 and s_n will occur nowhere in the collections of σ_j 's. Thus the entries in positions 1 and $n+1$ in $(n+1, n, \dots, 1)$ will remain unchanged.

(ii) In this case, $\sigma_1 = s_n s_{n-1} \cdots s_1$ and s_n will occur nowhere in $\sigma_2, \dots, \sigma_k$. So $\varphi(p) = \sigma_k \cdots \sigma_2(n, n-1, \dots, 1, n+1) = \sigma_k \cdots \sigma_2(n, n-1, \dots, 1), n+1 = (\varphi(p'), n+1)$.

(iii) Let us suppose that $p = NqEp'Nq^\perp E \in SS_n$ where $q \in S_{m-2}$, $p \in SS_{n-2m}$ and $q^\perp \in S_{m-2}$ is the reflection of q , where $m \leq \lfloor n/2 \rfloor$. Let $\sigma_1, \dots, \sigma_j$ be the strings of s_i 's constructed from the $Nq^\perp E$ region of p , $\sigma_{j+1}, \dots, \sigma_k$ the strings constructed from the p' region and $\sigma_{k+1}, \dots, \sigma_k + j$ the strings constructed from the NqE region. (Symmetry guarantees this latter collection has the same number of σ 's as the first.)

Let $\sigma' = \sigma_j \cdots \sigma_1$, $\sigma'' = \sigma_k \cdots \sigma_{j+1}$ and $\sigma''' = \sigma_{k+j} \cdots \sigma_{k+1}$. It is clear that $\sigma_1 = s_n \cdots s_{n-m+1}$ and $\sigma_{k+1} = \sigma_m \cdots \sigma_1$. Since the path has been partitioned into 3 distinct non-overlapping regions, σ' may operate only on positions $n-m+1$ to $n+1$ (inclusive) of a permutation, σ'' may operate only on positions $m+1$ to $n-m+1$ (inclusive) and σ''' may operate only on positions 1 to $m+1$ (inclusive.) This gives

$$\begin{aligned} \varphi(p) &= \sigma''' \sigma'' \sigma'(n+1, n, \dots, 1) \\ &= \sigma''' \sigma'' \sigma_j \cdots \sigma_2(n+1, \dots, m+2, m, \dots, 1, m+1) \\ &= \sigma''' \sigma''(n+1, \dots, m+2, a_1, \dots, a_m, m+1), \end{aligned}$$

where $a_i \in [1, m]$,

$$\begin{aligned} &= \sigma'''(n+1, \dots, n-m+2, b_1, \dots, b_{n-2m+1}, a_2, \dots, a_m, m+1) \\ &= \sigma_{k+j} \cdots \sigma_{k+2}(n, \dots, n-m+2, b_1, n+1, b_2, \dots, b_{n-2m+1}, a_2, \dots, a_m, m+1) \\ &= (c_1, \dots, c_m, n+1, b_2, \dots, b_{n-2m+1}, a_2, \dots, a_m, m+1). \end{aligned}$$

It is routine to check that this is the same as

$$(\sigma(qp'q^\perp) \leftarrow [1, \dots, m, m+2, \dots, n]) \cup (n+1, m+1)$$

where $\cup(n+1, m+1)$ signifies inserting $n+1$ between positions m and $m+1$ of the resulting sequences and inserting $m+1$ as a suffix (i.e. $(n+1, m+1)$ is a transposition in the resulting permutation.) \square

PROPOSITION 2.4. For all $p \in SS_n$ we have $\varphi(p) \in \mathcal{I}_{n+1}(1243, 2143)$.

PROOF. In [4] the map $\varphi : S_n \rightarrow \mathfrak{S}_{n+1}(1243, 2143)$ was shown to be a bijection. Thus we need only show that if $p \in SS_n$ then $\varphi(p)$ is an involution. This is easily done via induction. For the base

- If $(m+1, n+1)$ is a transposition of π for some $1 < m < n$, then $p = NqEp'Nq^\perp E$ where $q \in S_{m-1}$, $p' \in SS_{n-2m}$. Repeat the procedure for the permutation $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in I_{n-3}$ where

$$\alpha_i = \begin{cases} \pi_i & \text{if } i \leq m \text{ and } \pi_i < m+1, \\ \pi_i - 1 & \text{if } i \leq m \text{ and } \pi_i > m+1, \\ \pi_{i-1} & \text{if } i > m+1 \text{ and } \pi_i < m+1, \\ \pi_{i-1} - 1 & \text{if } i > m+1 \text{ and } \pi_i > m+1. \end{cases}$$

Note that once one has determined $r = P_1P_2 \dots$, where $P_i \in \{N, E, D\}$, such that $\varphi(r) = \alpha$ then q is the suffix of r which comprises of those steps ($\in \{N, E, D\}$) from $(0, 0)$ to $(m-1, m-1)$, p' is the subsequent sequence of steps from $(m-1, m-1)$ to $(n-m+1, n-m+1)$ and q^\perp is the remaining collection. □

2.2. The bijection $\Phi_2 : \mathcal{I}_n(1243, 2143) \mapsto \mathcal{I}_n(2134, 2143)$. Given $\pi \in \mathcal{I}_n(2134, 2143)$, let $\Phi_2(\pi) = (n+1-\pi_n, n+1-\pi_{n-1}, \dots, n+1-\pi_1)$. Clearly $\Phi_2 : \mathcal{I}_n(1243, 2143) \rightarrow \mathcal{I}_n(2134, 2143)$. It is a bijection since Φ_2 is the well known reverse-complement map and $\Phi_2^{-1} = \Phi_2$.

2.3. The bijection $\Phi_1 : \mathcal{I}_n(1234, 1243) \mapsto \mathcal{I}_n(2134, 2143)$. Given $\pi \in \mathcal{I}_n(1234, 1243)$ let us concentrate on occurrences of the patterns 2134, 2143. Let $A_\pi = \emptyset$. If $(\pi_{i(1)}, \pi_{i(2)}, \pi_{i(3)}, \pi_{i(4)})$ is an occurrence of the pattern 2134 in π then insert entries $i(1), i(2)$ into A_π . Do likewise for the pattern 2143. Let $A_\pi = \{i(1), i(2), \dots, i(t)\}$ be the resulting collection of all such indices.

LEMMA 2.7. *Given $\pi \in \mathcal{I}_n(1234, 1243)$ with sequence $A_\pi = \{i(1), i(2), \dots, i(t)\}$ as outlined above, $\pi_{i(1)} > \pi_{i(2)} > \dots > \pi_{i(t)}$.*

PROOF. Without loss of generality let us hinge our argument around the pattern 1234. Assume there exists $a < b$ such that $\pi_{i(a)} < \pi_{i(b)}$. The value $i(b)$ is in the sequence A_π for a reason: either there are indices a', c, d such that $a' < i(b) < c < d$ and $\pi_{a'}, \pi_{i(b)}, \pi_c, \pi_d$ is an occurrence of 2134 or $i(b) < a' < b < c$ and $\pi_{i(b)}, \pi_{a'}, \pi_c, \pi_d$ is an occurrence of 2134.

If $a' < i(b) < c < d$ and $\pi_{a'}, \pi_{i(b)}, \pi_c, \pi_d$ is an occurrence of 2134 then since $a < b$, $\pi_{i(a)}, \pi_{i(b)}, \pi_c, \pi_d$ is an occurrence of 1234 in π . This cannot be the case since π is 1234 avoiding.

So $i(b) < a' < b < c$ and $\pi_{i(b)}, \pi_{a'}, \pi_c, \pi_d$ is an occurrence of 2134. But since $a < b$, $i(a) < i(b)$ and $\pi_{i(a)}, \pi_{i(b)}, \pi_c, \pi_d$ will therefore be an occurrence of 1234. Again, this cannot be so since π is 1234 avoiding, hence have $\pi_{i(a)} > \pi_{i(b)}$. □

DEFINITION 2.8. *Let $\pi \in \mathcal{I}_n(1234, 1243)$. If π contains no occurrence of the patterns 2134 and 2143 then let $\Phi_1(\pi) = \pi$. Otherwise let $A_\pi = \{i(1), \dots, i(t)\}$ be the sequence associated with the 1 and 2's of occurrences of 2134 and 2143 in π as mentioned above. Let $\Phi_1(\pi) = \alpha_1 \dots \alpha_n$ where*

(1) $\alpha_j = \pi_j$ for all $j \notin A_\pi$, and

(2) let F be the largest Ferrers board (oriented as in Example 2.10) which is contained in the region strictly south west of every point of π not in A_π . The points $\{(i(1), \alpha_{i(1)}), \dots, (i(t), \alpha_{i(t)})\}$ are contained in F . In this board, recursively (in a lexicographic order) replace every occurrence of 21 by 12. Let $\{(i(1), \beta_{i(1)}), \dots, (i(t), \beta_{i(t)})\}$ be the outcome. Finally let $\alpha_{i(j)} = \beta_{i(j)}$ for all $1 \leq j \leq t$.

Note that in (2) an occurrence of 21 in F in this case means the smallest rectangle containing both points of the 21 pattern must also be contained in the Ferrers board F . This definition is a particular instance of the bijective map ϕ^* from Bousquet-Mélou and Steingrímsson [3]. The nature of the map is to iteratively replace occurrences of a monotone pattern in a Ferrers board so that upon termination there are no such occurrences.

PROPOSITION 2.9. *The map $\Phi_1 : \mathcal{I}_n(1234, 1243) \rightarrow \mathcal{I}_n(2134, 2143)$ is a bijection.*

PROOF. Let us first show the map Φ_1 is well defined. Given $\pi \in \mathcal{I}_n(1234, 1243)$, suppose, without loss of generality, there is an occurrence of 2134 in $\Phi_1(\pi) = \alpha_1 \dots \alpha_n$. We assume A_π is non-empty, for otherwise there would certainly be no occurrences of 2134 and 2143 in $\Phi_1(\pi)$ since $\Phi_1(\pi) = \pi$. Let us further suppose the lexicographically first occurrence of 2134 in $\Phi(\pi)$ is at (a, b, c, d) , i.e. $\alpha_a \alpha_b \alpha_c \alpha_d$ is such that $\alpha_b < \alpha_a < \alpha_c < \alpha_d$.

We note that, in this setting, at least one of $\{a, b, c, d\}$ must be contained in A_π , as otherwise one would have a 2134 pattern in π whose existence was not recorded.

If $d \in A_\pi = \{i(1), \dots, i(t)\}$ then it follows that $\pi^{-1}(\alpha_a), \pi^{-1}(\alpha_b), \pi^{-1}(\alpha_c) \in A_\pi$. Furthermore, since $\pi_{i(1)} > \dots > \pi_{i(t)}$, we must have that $\pi^{-1}(\alpha_a), \pi^{-1}(\alpha_b), \pi^{-1}(\alpha_c) > d$. Since operation (2) above was able to move the entries of α_a, α_b and α_c in π to the left of α_d in $\Phi_1(\pi)$, the rules of exchange on the Ferrers board would have replaced every occurrence of 21 by 12. This is not the case since $\alpha_a > \alpha_b$. Hence $d \notin A_\pi$. A similar argument shows that $c \notin A_\pi$.

The situation is that either (a) $a, b \in A_\pi$, (b) $a \in A_\pi, b \notin A_\pi$ or (c) $a \notin A_\pi, b \in A_\pi$.

For case (a), we have $\pi_a > \pi_b$ and the map Φ_1 will make $\alpha_a \geq \pi_a, \alpha_b \geq \pi_b, \alpha_c = \pi_c$ and $\alpha_d = \pi_d$. Note that $\alpha_b < \alpha_a < \alpha_c < \alpha_d$. Since $\alpha_a \alpha_b$ is an occurrence of 21 in $\Phi_1(\pi)$, there is some $j \notin A_\pi$ such that $a < j < b$ and $\pi_a > \pi_j > \pi_b$. The action of Φ_1 will ensure that $\alpha_a > \alpha_j > \alpha_b$. This means that $\alpha_a \alpha_j \alpha_c \alpha_d$ is an occurrence of 2134 in $\Phi_1(\pi)$ which is lexicographically smaller than $\alpha_a \alpha_b \alpha_c \alpha_d$. Hence case (a) is impossible.

For case (b), we have $a \in A_\pi, b, c, d \notin A_\pi$. Then $\alpha_a \geq \pi_a$ and $\alpha_a \pi_b \pi_c \pi_d$ is an occurrence of 2134 in $\Phi_1(\pi)$. But since $\alpha_a \geq \pi_a$ either $\pi_a > \pi_b$ or $\pi_a < \pi_b$. If $\pi_a > \pi_b$ then $\pi_a \pi_b \pi_c \pi_d$ is an occurrence of 2134 in π which would mean that both a and b are in A_π . Since this is not the case we must have that $\pi_a < \pi_b$ but this implies $\pi_a \pi_b \pi_c \pi_d$ is an occurrence of 1234 in π . Hence case (b) is impossible.

A similar argument shows (c) cannot be true either. Hence $\Phi_1(\pi)$ does not contain the pattern 2134. Similar reasoning shows this to be the case for the pattern 2143 also.

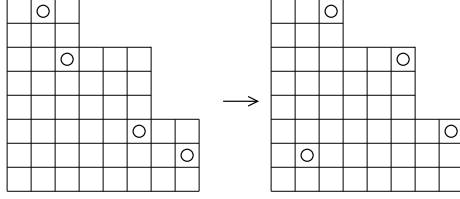
To show that Φ_1 is injective, the inverse bijection $f : \mathcal{I}_n(2134, 2143) \rightarrow \mathcal{I}_n(1234, 1243)$ of Φ_1 may be defined by: let $\pi \in \mathcal{I}_n(2134, 2143)$. If π contains no occurrence of the patterns 1234 and 1243 then let $f(\pi) = \pi$. Otherwise let $B_\pi = \{i(1), \dots, i(t)\}$ be the sequence associated with the 1 and 2's of occurrences of 1234 and 1243 in π . Let $f(\pi) = \beta_1 \dots \beta_n$ where

(1) $\beta_j = \pi_j$ if $j \notin B_\pi$ and

(2) let F be the largest Ferrers board which is contained in the region strictly south west of every point of π not in B_π . The points $\{(i(1), \beta_{i(1)}), \dots, (i(t), \beta_{i(t)})\}$ are contained in F . In this board, recursively (in a lexicographic order) replace every occurrence of 12 by 21. Let $\{(i(1), \gamma_{i(1)}), \dots, (i(t), \gamma_{i(t)})\}$ be the outcome. Now set $\beta_{i(j)} = \gamma_{i(j)}$ for all $1 \leq j \leq t$.

Surjectivity follows easily from Bousquet-Mélou and Steingrímsson [3, Theorem 1] since the sizes of $\mathcal{I}_n(1234, 1243)$ and $\mathcal{I}_n(2134, 2143)$ are equal. \square

EXAMPLE 2.10. Let $\pi = (11, 8, 6, 12, 10, 3, 7, 2, 9, 5, 1, 4)$. Then $A_\pi = \{2, 3, 6, 8\}$. Now the Ferrers board containing the 21's of π is transformed as:



This results in $\Phi_1(\pi) = (11, 2, 8, 12, 10, 6, 7, 3, 9, 5, 1, 4)$.

THEOREM 2.11. The map $f = \Phi_1^{-1} \circ \Phi_2 \circ \varphi : SS_{n-1} \mapsto \mathcal{I}_n(1234, 1243)$ is a bijection.

PROOF. From Proposition 2.6, Section 2.2 and Proposition 2.9 we have bijections $\varphi : SS_{n-1} \mapsto \mathcal{I}_n(1243, 2143)$, $\Phi_2 : \mathcal{I}_n(1243, 2143) \mapsto \mathcal{I}_n(2134, 2143)$ and $\Phi_1^{-1} : \mathcal{I}_n(2134, 2143) \mapsto \mathcal{I}_n(1234, 1243)$. The composition $f = \Phi_1^{-1} \circ \Phi_2 \circ \varphi$ is therefore a bijection $f : SS_{n-1} \mapsto \mathcal{I}_n(1234, 1243)$. \square

We note that it would be nice to have a direct bijection between the set $\mathcal{I}_{n+1}(1234, 1243)$ of involutions of length n avoiding both 1234 and 1243, and the set SS_n of Schröder paths of length n , or even between the set of permutations $\mathfrak{S}_{n+1}(1234, 1243)$ and the set S_n of Schröder paths of length n .

3. Proof of Theorem 1.1

To present the proof of Theorem 1.1, we must first consider the enumeration problem for the number \mathcal{F}_k -avoiding involutions according to length and number of fixed points, where \mathcal{F}_k is the set of all permutations $\sigma \in \mathfrak{S}_k$ with $\sigma_1 = 1$.

3.1. Involutions avoiding \mathcal{F}_k . In this subsection we present an explicit formula for the number of involutions that avoid all the patterns in \mathcal{F}_k . To do so we require some new notation. Define $f_k(n)$ to be the number of involutions $\pi \in \mathcal{I}_n(\mathcal{F}_k)$. Given $t \in [n]$, we also define

$$f_{k;m}(n; t) = \#\{\pi \in \mathcal{I}_n(\mathcal{F}_k) \mid \pi_1 = t \text{ and } \pi \text{ contains } m \text{ fixed points}\}.$$

Let $f_k(n; t) = f_k(n, p; t)$ and $f_k(n) = f_k(n, p)$ be the polynomials $\sum_{m=0}^n f_{k;m}(n; t)p^m$ and $\sum_{t=1}^n f_k(n; t)$, respectively. We denote by $F_k(x, p)$ the generating function for the sequence $f_k(n, p)$, that is $F_k(x, p) = \sum_{n \geq 0} f_k(n, p)x^n$.

THEOREM 3.1. We have

$$F_k(x, p) = \sum_{j=0}^{k-2} J_j(p)x^j + \frac{x^{k-1}}{1 - (k-1)x^2}((k-1)J_{k-2}(p)x + J_{k-1}(p)).$$

Moreover, the number of involutions of length $k + 2n$ (resp. $k + 2n - 1$) that avoid all the patterns in \mathcal{F}_k is given by $(k-1)^{n+1}I_{k-2}$ (resp. $(k-1)^n I_{k-1}$), for all $n \geq 0$.

PROOF. Let $\pi \in \mathfrak{S}_n$ be a permutation that avoids all patterns in \mathcal{F}_k . We have $\pi_1 \geq n + 2 - k$. Thus $\pi \in \mathcal{I}_n(\mathcal{F}_k)$ with $\pi_1 = t \geq n + 2 - k$ if and only if $\pi_2 \dots \pi_{t-1} \pi_{t+1} \dots \pi_n$ is an involution on the numbers $2, \dots, t-1, t+1, \dots, n$ that avoids all the patterns in \mathcal{F}_k . Hence, $f_k(n; j) = f_k(n-2)$ for

all $j = n + 2 - k, n + 3 - k, \dots, n$, and $f_k(n, j) = 0$ for all $j = 1, 2, \dots, n + 1 - k$, where $n \geq k$. Thus, for $n \geq k$,

$$f_k(n) = (k - 1)f_k(n - 2).$$

Using the initial conditions $f_k(j) = J_j(p)$, $j = 1, 2, \dots, k - 1$, we find that $f_k(k + 2j) = (k - 1)^{j+1}J_{k-1}(p)$ and $f_k(k + 2j - 1) = (k - 1)^j J_{k-2}(p)$ for all $j \geq 0$. Rewriting these formulas in terms of generating functions we obtain

$$F_k(x, p) = \sum_{j=0}^{k-2} J_j(p)x^j + \frac{x^{k-1}}{1 - (k-1)x^2}((k-1)J_{k-2}(p)x + J_{k-1}(p)),$$

as claimed. \square

3.2. Involution avoiding \mathcal{A}_k . In this subsection we prove Theorem 1.1. In order to do this, define $g_k(n)$ to be the number of involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ and given $t_1, t_2, \dots, t_m \in \mathbb{N}$, we also define

$$g_k(n; t_1, t_2, \dots, t_m) = \#\{\pi_1 \dots \pi_n \in \mathcal{I}_n(\mathcal{A}_k) \mid \pi_1 \dots \pi_m = t_1 \dots t_m\}.$$

LEMMA 3.2. *Let $k \geq 3$. For all $3 \leq t \leq n + 1 - k$,*

$$g_k(n; t) = (k - 2)g_k(n - 2; t - 1) + \sum_{j=1}^{t-2} g_k(n - 2; j),$$

with the initial conditions $g_k(n; 1) = f_{k-1}(n - 1)$, $g_k(n; 2) = f_{k-1}(n - 2)$, and $g_k(n; t) = g_k(n - 2)$ for all $t = n + 2 - k, n + 3 - k, \dots, n$.

PROOF. Let π be any involution of length n that avoids all patterns in \mathcal{A}_k with $\pi_1 = t$. Now let us consider all possible values of t . If $t = 1$ then $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ if and only if $(\pi_2 - 1)(\pi_3 - 1) \dots (\pi_n - 1) \in \mathcal{I}_{n-1}(\mathcal{F}_{k-1})$. If $t = 2$ then $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ if and only if $(\pi_3 - 2)(\pi_4 - 2) \dots (\pi_n - 2) \in \mathcal{I}_{n-2}(\mathcal{F}_{k-1})$. Now assume that $3 \leq t \leq n + 1 - k$, then from the above definitions

$$g_k(n; t) = g_k(n; t, 1) + \dots + g_k(n; t, t - 1) + g_k(n; t, t + 1) + \dots + g_k(n; t, n).$$

But any involution π satisfying $\pi_1 < \pi_2 \leq n + 2 - k$ contains a pattern from the set \mathcal{A}_k (see the subsequence of the letters $\pi_1, \pi_2, n + 3 - k, n + 4 - k, \dots, n$ in π). Thus $g_k(n; t, r) = 0$ for all $t < r \leq n + 2 - k$ and so

$$g_k(n; t) = g_k(n; t, 1) + \dots + g_k(n; t, t - 1) + g_k(n; t, n + 3 - k) + \dots + g_k(n; t, n).$$

Also, if π is an involution in \mathcal{I}_n with $\pi_1 = t$ and $\pi_2 = r \geq n + 3 - k$, then the entry r does not appear in any occurrence of $\tau \in \mathcal{A}_k$ in π . Thus, there exists a bijection between the set of involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ with $\pi_1 = t$ and $\pi_2 = r \geq n + 3 - k$ and the set of involutions $\pi' \in \mathcal{I}_{n-2}(\mathcal{A}_k)$ with $\pi' = t - 1$. Therefore $g_k(n; t, r) = g_k(n - 2; t - 1)$ which gives

$$g_k(n; t) = g_k(n; t, 1) + \dots + g_k(n; t, t - 1) + (k - 2)g_k(n - 2; t - 1).$$

Also, if π is an involution in \mathcal{I}_n with $\pi_1 = t$, $\pi_2 = r < t$ and if $ta_2 \dots a_k$ is an occurrence of a pattern from the set \mathcal{A}_k in π , then $ra_2 \dots a_k$ is an occurrence of a pattern from the set \mathcal{A}_k in π . Thus, there exists a bijection between the set of involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ with $\pi_1 = t$ and $\pi_2 = r < t$ and the set of involutions $\pi' \in \mathcal{I}_{n-2}(\mathcal{A}_k)$ with $\pi'_1 = r - 1$. Therefore $g_k(n; t, r) = g_k(n - 2; r - 1)$ which gives

$$g_k(n; t) = (k - 2)g_k(n - 2; t - 1) + \sum_{j=1}^{t-2} g_k(n - 2; j),$$

as required. Finally, if π is an involution in \mathcal{I}_n with $\pi_1 = t \geq n + 2 - k$, then the entry t does not appear in any occurrence of $\tau \in \mathcal{A}_k$ in π . Thus, there exists a bijection between the set of

involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ with $\pi_1 = t \geq n + 2 - k$ and the set of involutions $\pi' \in \mathcal{I}_{n-2}(\mathcal{A}_k)$. Therefore $g_k(n; t) = g_k(n - 2)$, as claimed. \square

Let $G_k(n; v)$ be the polynomial $\sum_{t=1}^n g_k(n; t)v^{t-1}$. Rewriting the above lemma in terms of the polynomials $G_k(n; v)$ we have the following recurrence relation.

LEMMA 3.3. *Let $k \geq 3$. For all $n \geq k$,*

$$G_k(n; v) = f_{k-1}(n-1) + v f_{k-1}(n-2) - v(k-2)f_{k-1}(n-3) \\ + \left(\frac{v^2}{1-v} + (k-2)v \right) G_k(n-2; v) - \frac{v^n}{1-v} G_k(n-2; 1) + \frac{v^{n-1}}{1-v} \left(k-2 + \frac{v-v^{3-k}}{1-v} \right) G_k(n-4; 1),$$

where $G_k(n; v) = I_{n-1} + \frac{v-v^n}{1-v} I_{n-2}$ for all $n = 0, 1, \dots, k-1$.

PROOF. Lemma 3.2 gives

$$G_k(n; v) = f_{k-1}(n-1) + v f_{k-1}(n-2) + \sum_{t=2}^{n-k} v^t \left((k-2)G_k(n-2; t) + \sum_{j=1}^{t-1} G_k(n-2; j) \right) \\ + \left(\sum_{t=n+1-k}^{n-1} v^t \right) G_k(n-2; 1) \\ = f_{k-1}(n-1) + v f_{k-1}(n-2) + \frac{v^2}{1-v} (G_k(n-2; v) - G_k(n-4; 1) \sum_{j=n-1-k}^{n-3} v^j) \\ - \frac{v^{n+1-k}}{1-v} (G_k(n-2; 1) - (k-1)G_k(n-4; 1)) + \frac{v^{n+1-k}-v^n}{1-v} G_k(n-2; 1) \\ + (k-2)v \left(G_k(n-2; v) - f_{k-1}(n-3) - G_k(n-4; 1) \sum_{j=n-k}^{n-3} v^j \right),$$

which is equivalent to

$$G_k(n; v) = f_{k-1}(n-1) + v f_{k-1}(n-2) - v(k-2)f_{k-1}(n-3) \\ + \left(\frac{v^2}{1-v} + (k-2)v \right) G_k(n-2; v) - \frac{v^n}{1-v} G_k(n-2; 1) + \frac{v^{n-1}}{1-v} \left(k-2 + \frac{v-v^{3-k}}{1-v} \right) G_k(n-4; 1).$$

To find the value of $G_k(n; v)$ for $n \leq k-1$, let π be any involution with $\pi_1 = t$. If $t = 1$ then there are I_{n-1} involutions, whereas if $t > 1$ there are I_{n-2} involutions, hence $G_k(n; v) = v^0 I_{n-1} + \sum_{t=2}^n v^{t-1} I_{n-2} = I_{n-1} + \frac{v-v^n}{1-v} I_{n-2}$, as required. \square

Lemma 3.3 can be generalised as follows; let $g_{k;m}(n; t)$ be the number of involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ such that $\pi_1 = t$ and π contains exactly m fixed points. Define $G_k(n; t; p) = \sum_{m=0}^n g_{k;m}(n; t)p^m$ and $G_k(n; v; p) = \sum_{t=1}^n G_k(n; t; p)v^{t-1}$. Using the same arguments as those in the proofs of Lemma 3.2 and Lemma 3.3, while carefully considering the number of fixed points, we have the following result.

LEMMA 3.4. *Let $k \geq 3$. For all $n \geq k$,*

$$G_k(n; v, p) \\ = p f_{k-1}(n-1) + v f_{k-1}(n-2) - p v (k-2) f_{k-1}(n-3) + \left(\frac{v^2}{1-v} + (k-2)v \right) G_k(n-2; v, p) \\ - \frac{v^n}{1-v} G_k(n-2; 1, p) + \frac{v^{n-1}}{1-v} \left(k-2 + \frac{v-v^{3-k}}{1-v} \right) G_k(n-4; 1, p),$$

where $G_k(n; v, p) = p J_{n-1}(p) + \frac{v-v^n}{1-v} J_{n-2}(p)$ for all $n = 0, 1, \dots, k-1$.

Let $G_k(x, v, p) = \sum_{n \geq 0} G_k(n; v, p)x^n$ be the generating function for the sequence $G_k(n; v, p)$. Define $J_i(v, p)$ to be the polynomial $\sum d_{tr} v^t p^r$ where d_{tr} is the number of involutions $\pi \in \mathcal{I}_i$ such that

$\pi_1 = t + 1$ and π contains exactly r fixed points. Rewriting the recurrence relation in the statement of Lemma 3.4 in terms of generating functions we obtain

$$\begin{aligned} G_k(x, v, p) &= \sum_{j=0}^{k-1} J_j(v, p)x^j + px \left(F_{k-1}(x, p) - \sum_{j=0}^{k-2} J_j(p)x^j \right) + vx^2 \left(F_{k-1}(x, p) - \sum_{j=0}^{k-3} J_j(p)x^j \right) \\ &\quad - (k-2)pvx^3 \left(F_{k-1}(x, p) - \sum_{j=0}^{k-4} J_j(p)x^j \right) - \frac{v^2x^2}{1-v} \left(G_k(xv, 1, p) - \sum_{j=0}^{k-3} J_j(p)(xv)^j \right) \\ &\quad + vx^2 \left(\frac{v}{1-v} + k-2 \right) \left(G_k(x, v, p) - \sum_{j=0}^{k-3} J_j(v, p)x^j \right) \\ &\quad + \frac{(k-2)v^3x^4}{1-v} \left(G_k(xv, 1, p) - \sum_{j=0}^{k-5} J_j(p)(xv)^j \right) - \frac{x^4(1-v^{k-2})}{v^{k-6}(1-v)^2} \left(G_k(xv, 1, p) - \sum_{j=0}^{k-5} J_j(p)(xv)^j \right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\left(1 - \frac{x^2}{1-v} - (k-2)\frac{x^2}{v} \right) G_k(x/v, v, p) \\ &= -\frac{x^2}{1-v} \left(1 - (k-2)\frac{x^2}{v} + \frac{x^2(1-v^{k-2})}{v^{k-2}(1-v)} \right) G_k(x, 1, p) \\ &\quad + \sum_{j=0}^{k-1} J_j(v, p)\frac{x^j}{v^j} + \frac{vx}{v} \left(F_{k-1}(x/v, p) - \sum_{j=0}^{k-2} J_j(p)\frac{x^j}{v^j} \right) + \frac{x^2}{v} \left(F_{k-1}(x/v, p) - \sum_{j=0}^{k-3} J_j(p)\frac{x^j}{v^j} \right) \\ &\quad - (k-2)p\frac{x^3}{v^2} \left(F_{k-1}(x/v, p) - \sum_{j=0}^{k-4} J_j(p)\frac{x^j}{v^j} \right) + \frac{x^2}{1-v} \sum_{j=0}^{k-3} J_j(p)x^j \\ &\quad - \frac{x^2}{v} \left(\frac{v}{1-v} + k-2 \right) \sum_{j=0}^{k-3} J_j(v, p)\frac{x^j}{v^j} - \frac{(k-2)x^4}{v(1-v)} \sum_{j=0}^{k-5} J_j(p)x^j + \frac{x^4(1-v^{k-2})}{v^{k-2}(1-v)^2} \sum_{j=0}^{k-5} J_j(p)x^j. \end{aligned}$$

To solve this functional equation, we substitute

$$v := v_0 = \frac{1}{2} \left(1 + (k-3)x^2 + \sqrt{1 - 2(k-1)x^2 + (k-3)^2x^4} \right),$$

where v_0 is the root of the coefficient of $G_k(x/v, v, p)$ above, into the above functional equation, that is, $1 - \frac{x^2}{1-v_0} - (k-2)\frac{x^2}{v_0} = 0$. Since $J_j(v, p) = pJ_{j-1}(p) + \frac{v-v^j}{1-v}J_{j-2}(p)$ for all $j = 1, 2, \dots, k-1$ and $J_0(v, p) = 1$, it is routine to show (via some rather tedious algebraic manipulation) that we obtain Theorem 1.1.

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