

Ehrhart polynomials of lattice-face polytopes

by *Fu Liu*

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June 22, 2006

Preliminaries

Definition 1. A *(convex) polytope* P in the d -dimensional Euclidean space \mathbb{R}^d is the convex hull of finitely many points $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \text{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n : \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \dots + \lambda_n = 1\}.$$

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Definition 2. For any polytope $P \subset \mathbb{R}^d$ and some positive integer $m \in \mathbb{N}$, the *m th dilated polytope* of P is $mP = \{m\mathbf{x} : \mathbf{x} \in P\}$. We denote by

$$i(m, P) = |\mathcal{L}(mP)|$$

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Example: When $d = 1$, P is an interval $[a, b]$, where $a, b \in \mathbb{Z}$. Then $mP = [ma, mb]$ and

$$i(P, m) = (b - a)m + 1.$$

Theorem of Ehrhart

Theorem 3. (Ehrhart) *Let P be a d -dimensional integral polytope, then $i(P, m)$ is a polynomial in m of degree d .*

Therefore, we call $i(P, m)$ the *Ehrhart polynomial* of P .

Coefficients of Ehrhart polynomials

If P is an integral polytope, what we can say about the coefficients of its Ehrhart polynomial $i(P, m)$?

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- ▣▶ The second coefficient equals $1/2$ times the sum of volumes of each facet, each normalized with respect to the sublattice in the hyperplane spanned by the facet.
- ▣▶ The constant term of $i(P, m)$ is always 1.
- ▣▶ No results for other coefficients for general polytopes.

Motivation

De Loera conjectured that the Ehrhart polynomial of an integral cyclic polytope has a simple formula.

Recall that given $T = \{t_1, \dots, t_n\}$ a linearly ordered set, a d -dimensional *cyclic polytope* $C_d(T) = C_d(t_1, \dots, t_n)$ is the convex hull $\text{conv}\{\nu_d(t_1), \nu_d(t_2), \dots, \nu_d(t_n)\}$ of $n > d$ distinct points $\nu_d(t_i)$, $1 \leq i \leq n$, on the moment curve.

The *moment curve* in \mathbb{R}^d is defined by

$$\nu_d : \mathbb{R} \rightarrow \mathbb{R}^d, t \mapsto \nu_d(t) = \begin{pmatrix} t \\ t^2 \\ \vdots \\ t^d \end{pmatrix}.$$

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Example: $T = \{1, 2, 3, 4\}$, $d = 3$:

$C_d(T)$ is the convex polytope whose vertices are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \\ 27 \end{pmatrix}, \begin{pmatrix} 4 \\ 16 \\ 64 \end{pmatrix}$.

Theorem 4. For any d -dimensional integral cyclic polytope $C_d(T)$,

$$i(C_d(T), m) = \text{Vol}(mC_d(T)) + i(C_{d-1}(T), m).$$

Hence,

$$\begin{aligned} i(C_d(T), m) &= \sum_{k=0}^d \text{Vol}_k(mC_k(T)) \\ &= \sum_{k=0}^d \text{Vol}_k(C_k(T))m^k, \end{aligned}$$

where $\text{Vol}_k(mC_k(T))$ is the volume of $mC_k(T)$ in k -dimensional space, and by convention we let $\text{Vol}_0(mC_0(T)) = 1$.

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\Rightarrow 2, 4, 3 and 1 are the volumes of $C_3(T)$, $C_2(T)$, $C_1(T)$ and $C_0(T)$, respectively.

Note that if we define $\pi^k : \mathbb{R}^d \rightarrow \mathbb{R}^{d-k}$ to be the map which ignores the last k coordinates of a point, then $\pi^k(C_d(T)) = C_{d-k}(T)$. So when $P = C_d(T)$ is an integral cyclic polytope, we have that

$$i(P, m) = \text{Vol}(mP) + i(\pi(P), m) = \sum_{k=0}^d \text{Vol}_k(\pi^{d-k}(P))m^k, \quad (5)$$

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Question: Are there other integral polytopes which have the same form of Ehrhart polynomials as cyclic polytopes? In other words, what kind of integral d -polytopes P are there whose Ehrhart polynomials will be in the form of (5)?

Properties of integral cyclic polytopes

What are some key properties of an integral cyclic polytope $C_d(T)$?

When $d = 1$, $C_d(T)$ is just an integral polytope.

For $d \geq 2$, for any d -subset $T' \subset T$, let $U = \nu_d(T')$ be the corresponding d -subset of the vertex set $V = \nu_d(T)$ of $C_d(T)$. Then:

- a) $\pi(\text{conv}(U)) = \pi(C_d(T')) = C_{d-1}(T')$ is an integral cyclic polytope, and
- b) $\pi(\mathcal{L}(H_U)) = \mathbb{Z}^{d-1}$, where H_U is the affine space spanned by U . In other words, after dropping the last coordinate of the lattice of H_U , we get the $(d - 1)$ -dimensional lattice.

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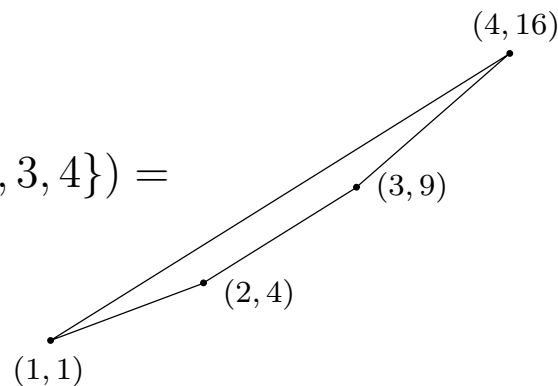
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$$P = C_2(\{1, 2, 3, 4\}) =$$



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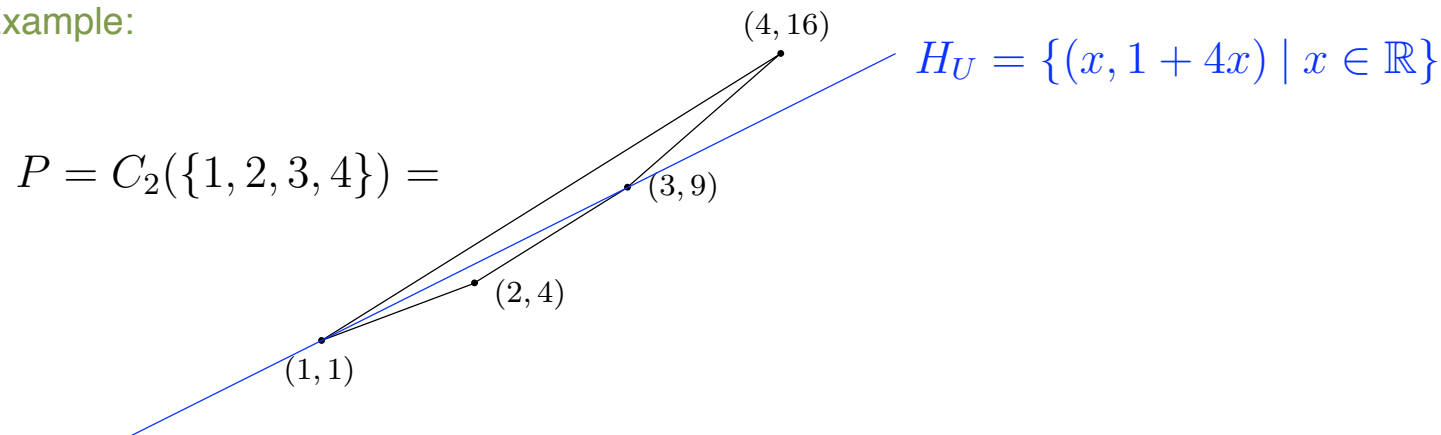
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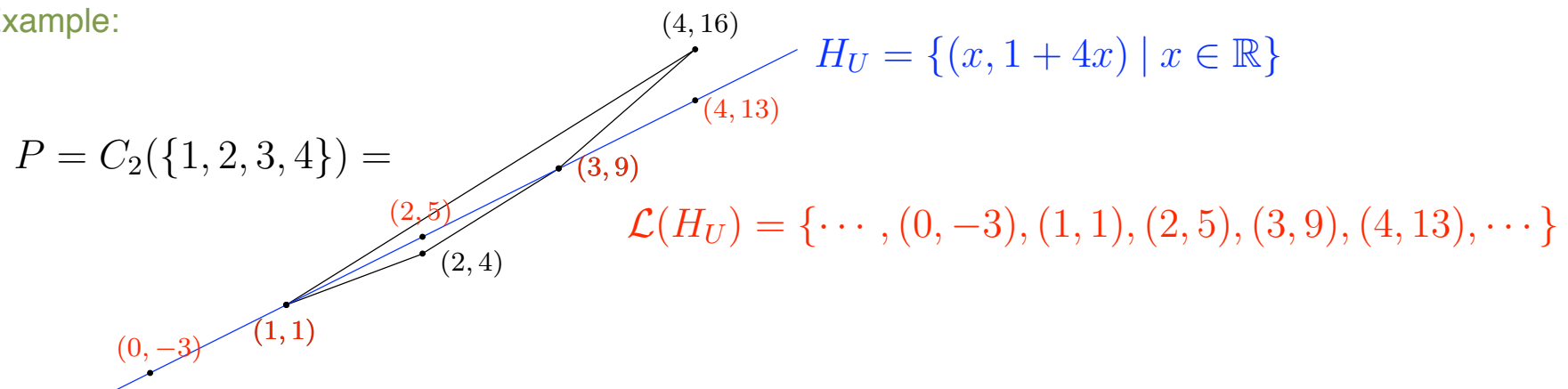
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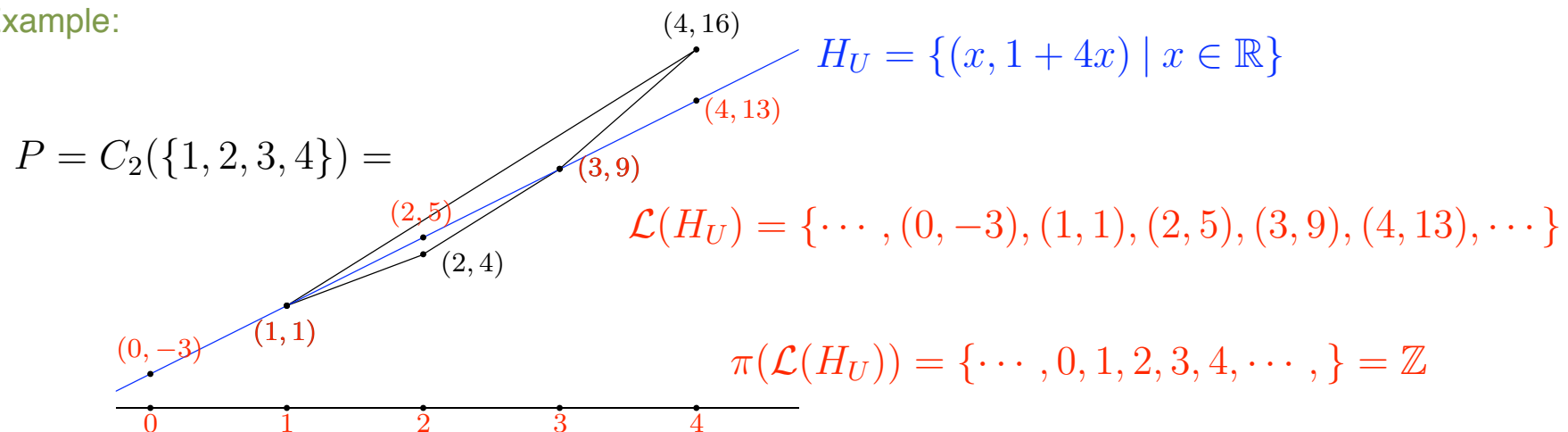
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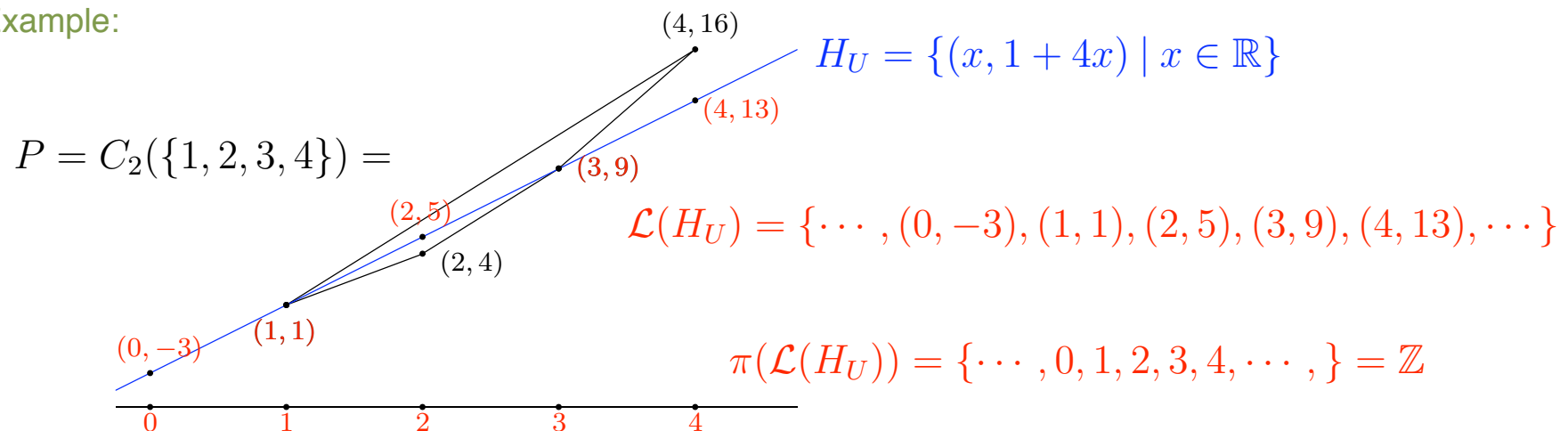
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Remark: Condition b) is equivalent to say that for any lattice point $y \in \mathbb{Z}^{d-1}$, we have that

$\pi^{-1}(y) \cap H_U$, the intersection of H_U with the inverse image of y under π , is a lattice point.

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The Main Theorem

Theorem 8. *Let P be a lattice-face d -polytope, then*

$$i(P, m) = \text{Vol}(mP) + i(\pi(P), m) = \sum_{k=0}^d \text{Vol}_k(\pi^{d-k}(P)) m^k.$$

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2. mP is a lattice-face d -polytope \Rightarrow it's enough to show that

$$|\mathcal{L}(P)| = \text{Vol}(P) + |\mathcal{L}(\pi(P))|.$$

More Notation

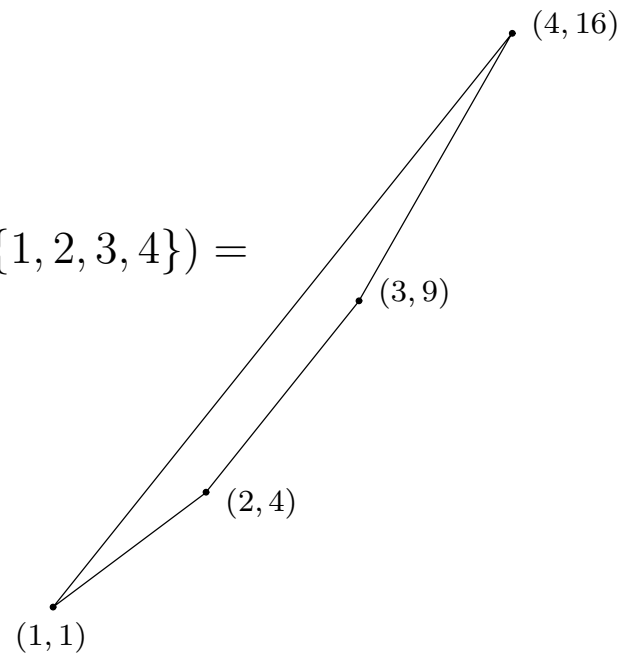
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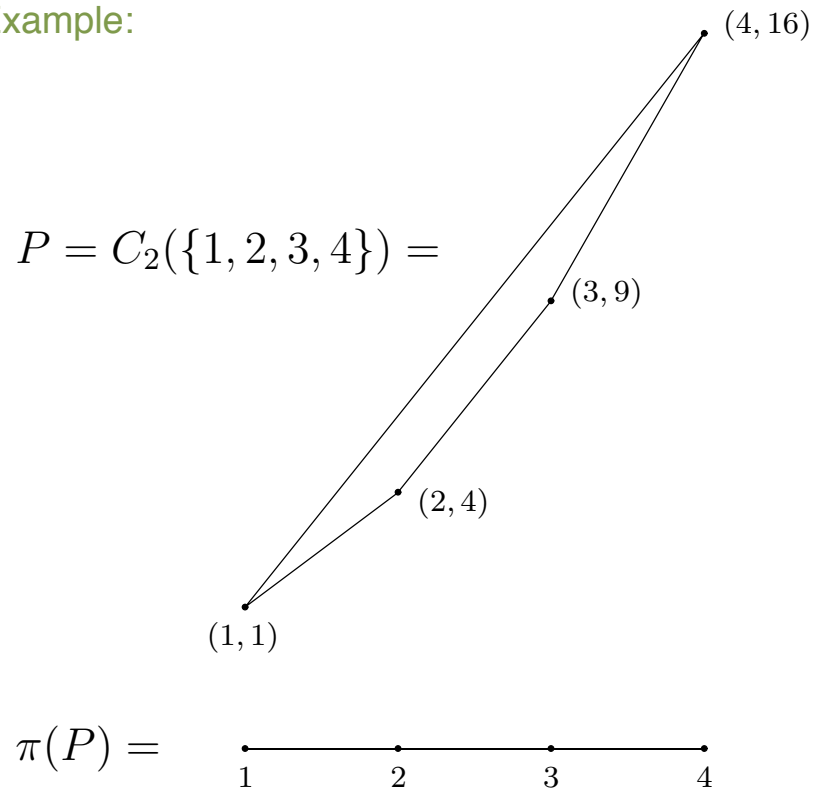
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1. For any polytope $P \subset \mathbb{R}^d$ and any point $y \in \pi(P)$, let $n(y, P)$ be the point of $\pi^{-1}(y) \cap P$ having the smallest last coordinate.
2. Define $NB(P) = \cup_{y \in \pi(P)} n(y, P)$ to be the *negative boundary* of P and $\Omega(P) = P \setminus NB(P)$ to be the *nonnegative part* of P .

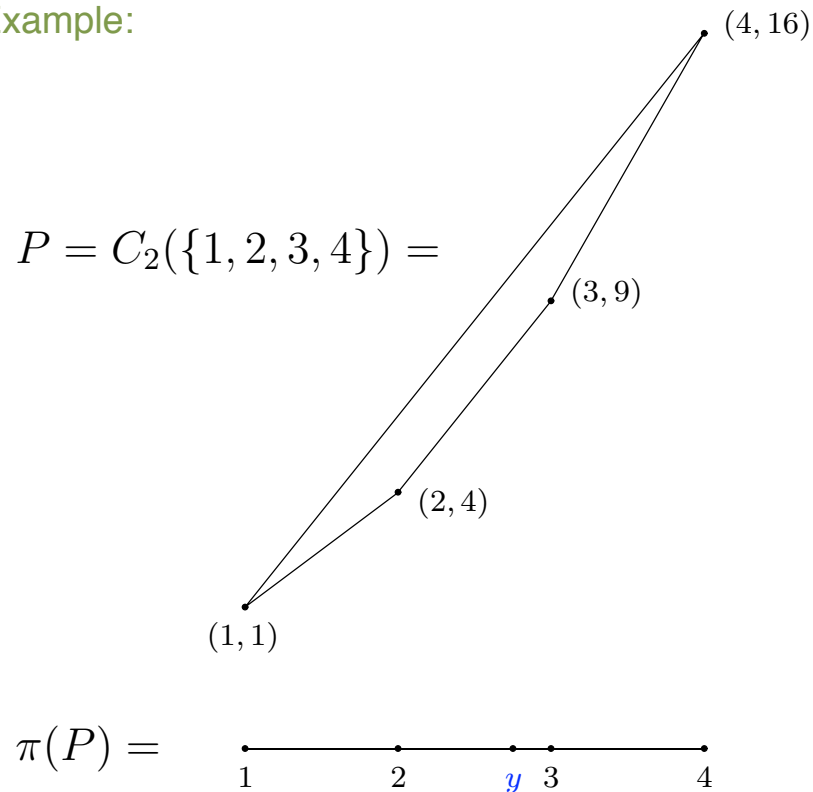
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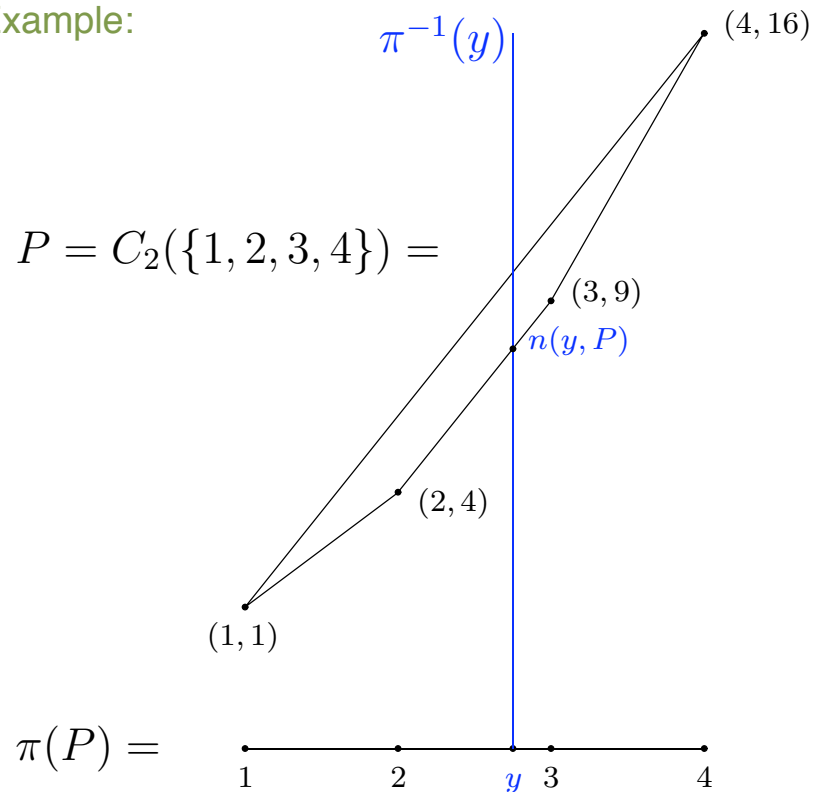
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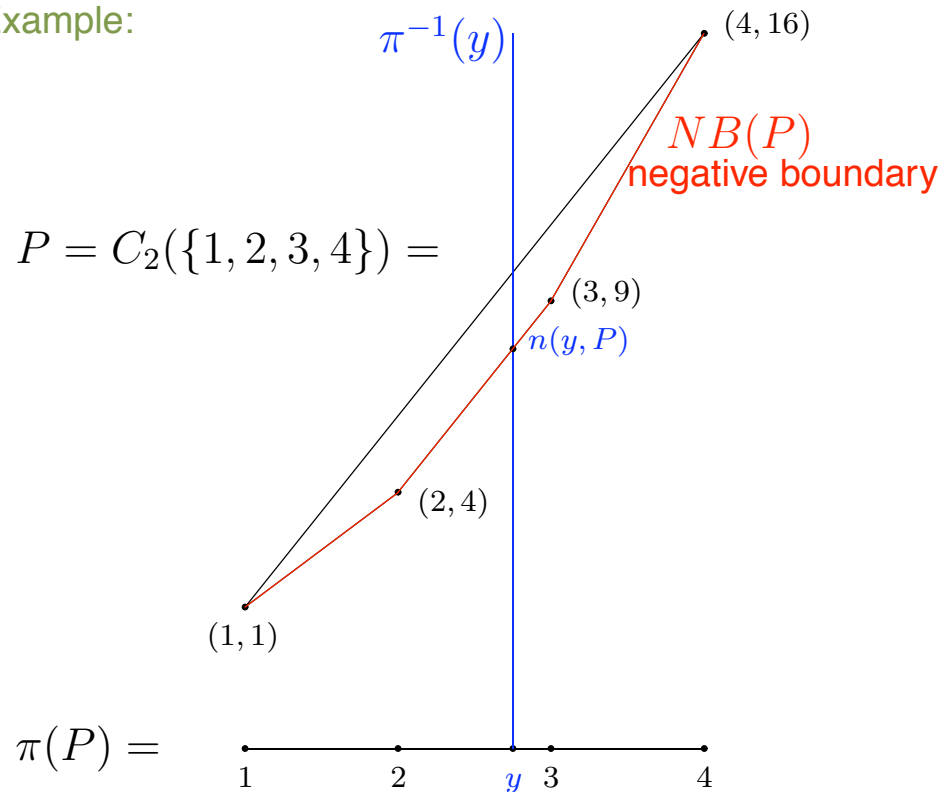
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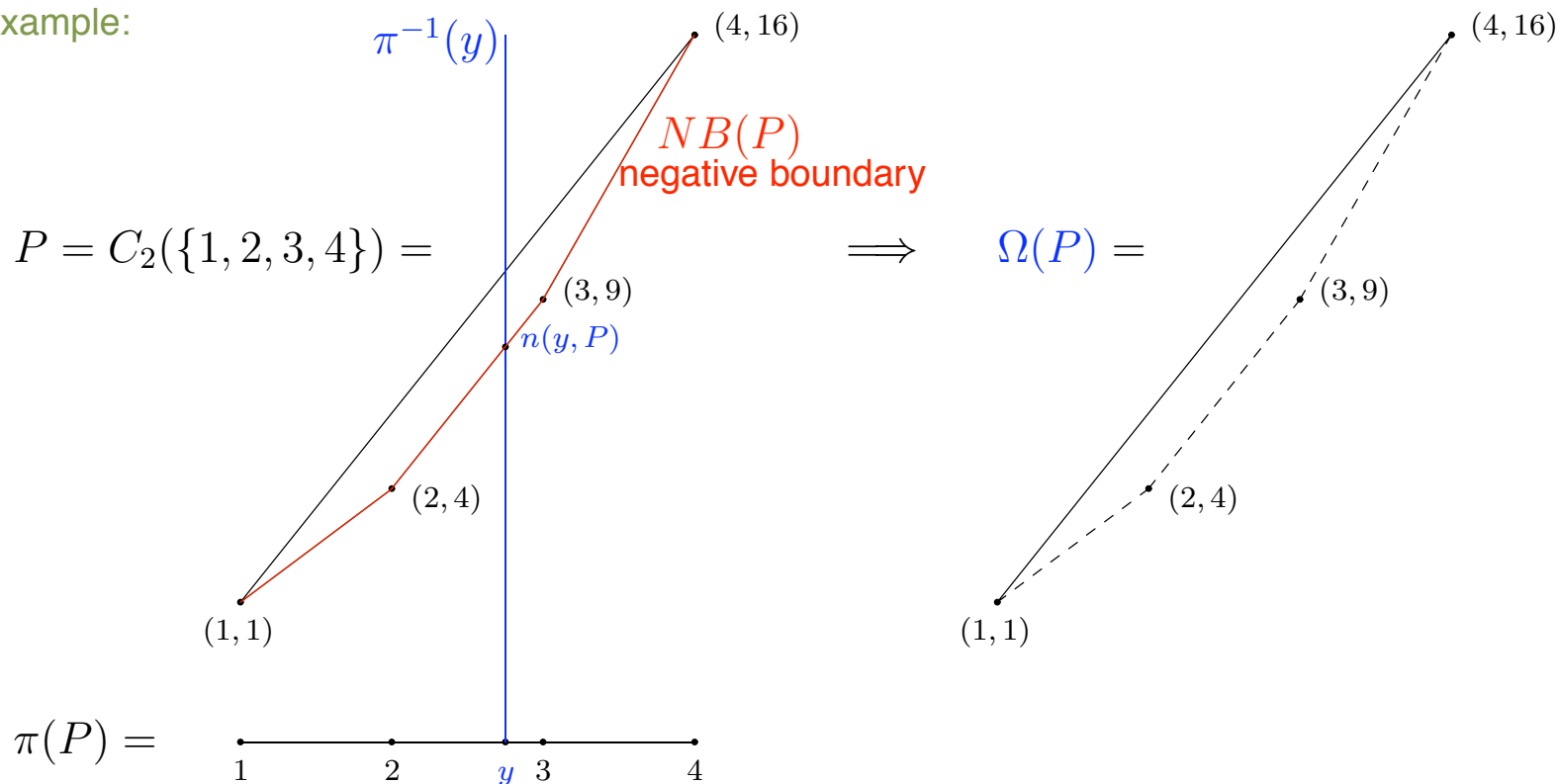
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Example:



Clearly, π induces a bijection between $\mathcal{L}(NB(P))$ and $\mathcal{L}(\pi(P))$. Therefore,

$$\begin{aligned} & |\mathcal{L}(P)| \\ &= |\mathcal{L}(\Omega(P))| + |\mathcal{L}(NB(P))| \\ &= |\mathcal{L}(\Omega(P))| + |\mathcal{L}(\pi(P))|. \end{aligned}$$

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Comparing with the formula we want to show:

$$|\mathcal{L}(P)| = \text{Vol}(P) + |\mathcal{L}(\pi(P))|,$$

one see that to prove Theorem 8 it is sufficient to prove the following theorem.

Theorem 9. *For any P a lattice-face d -polytope,*

$$\text{Vol}(P) = |\mathcal{L}(\Omega(P))|.$$

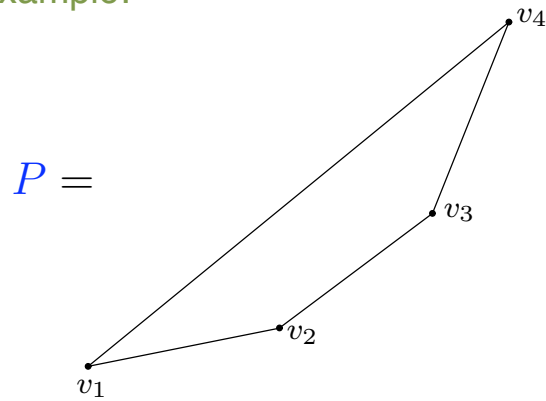
For any triangulation (without introducing new vertices) $P_1 \cup \cdots \cup P_k$ of a lattice-face polytope P , (note that (iv) implies that all P_i are lattice-face polytopes,) we have that

$$\Omega(P) = \bigoplus_{i=1}^k \Omega(P_i), \text{ which implies that } |\mathcal{L}(\Omega(P))| = \sum_{i=1}^k |\mathcal{L}(\Omega(P_i))|.$$

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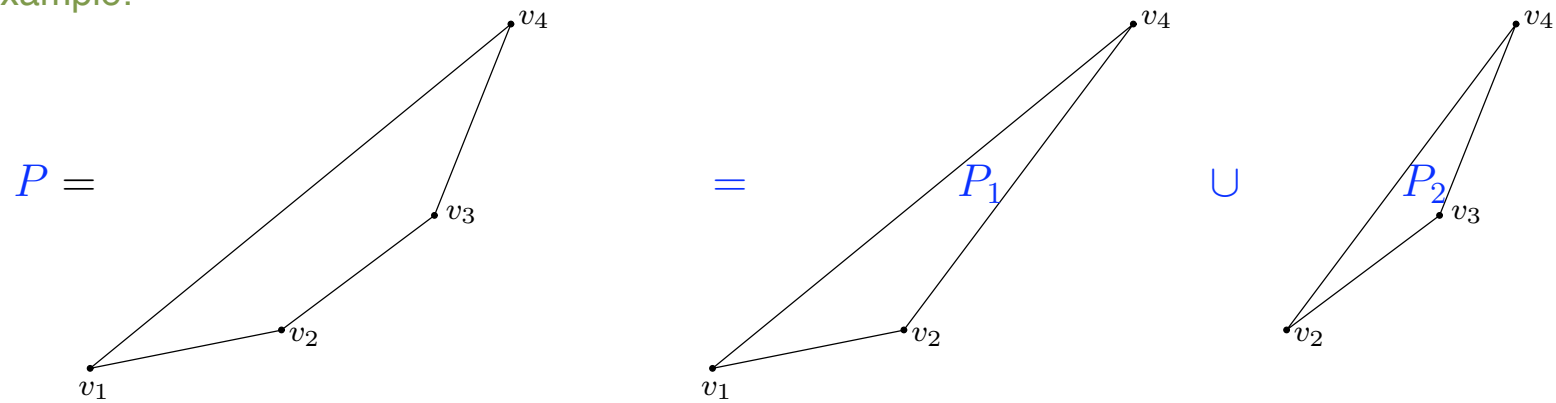
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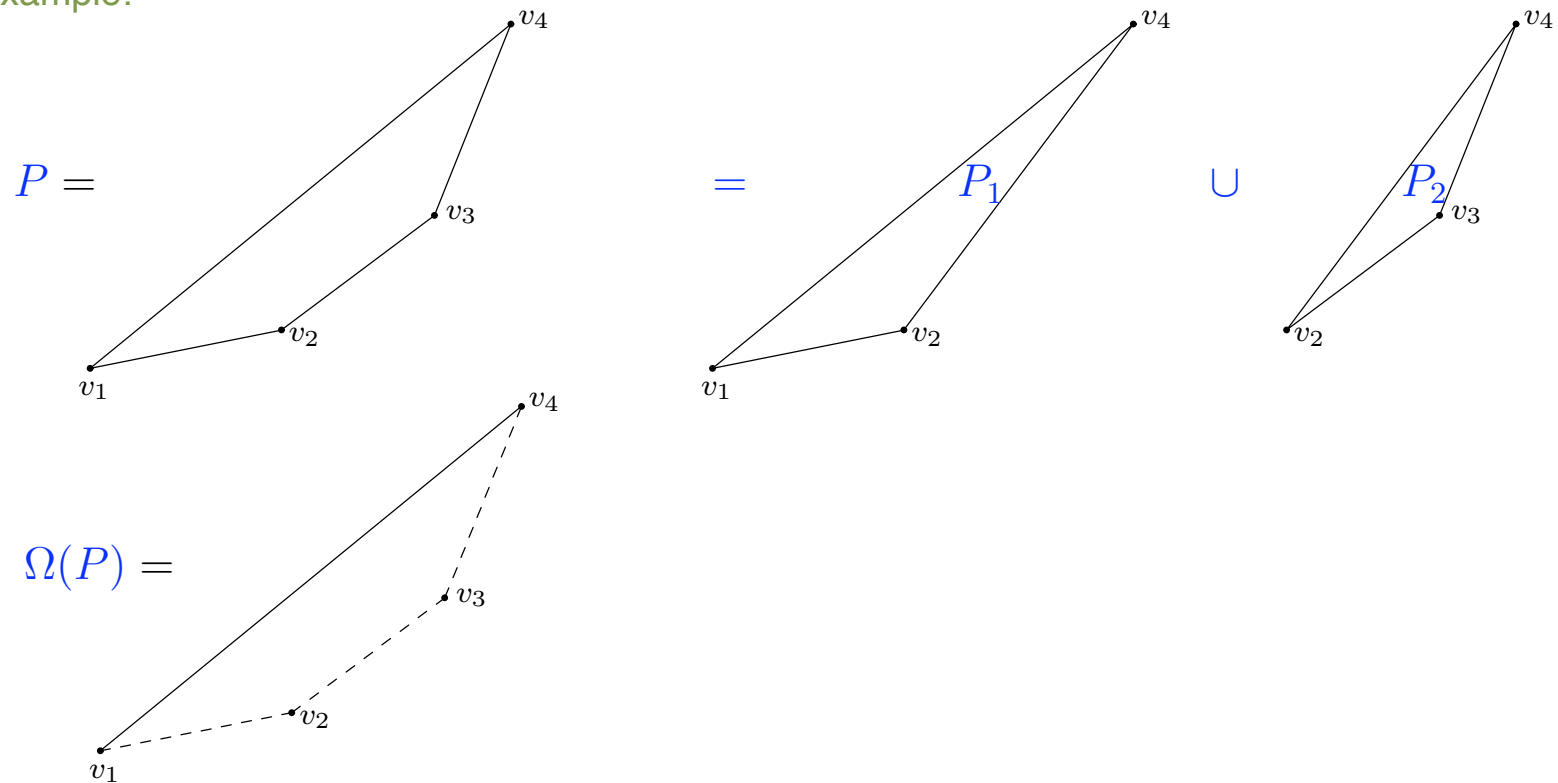
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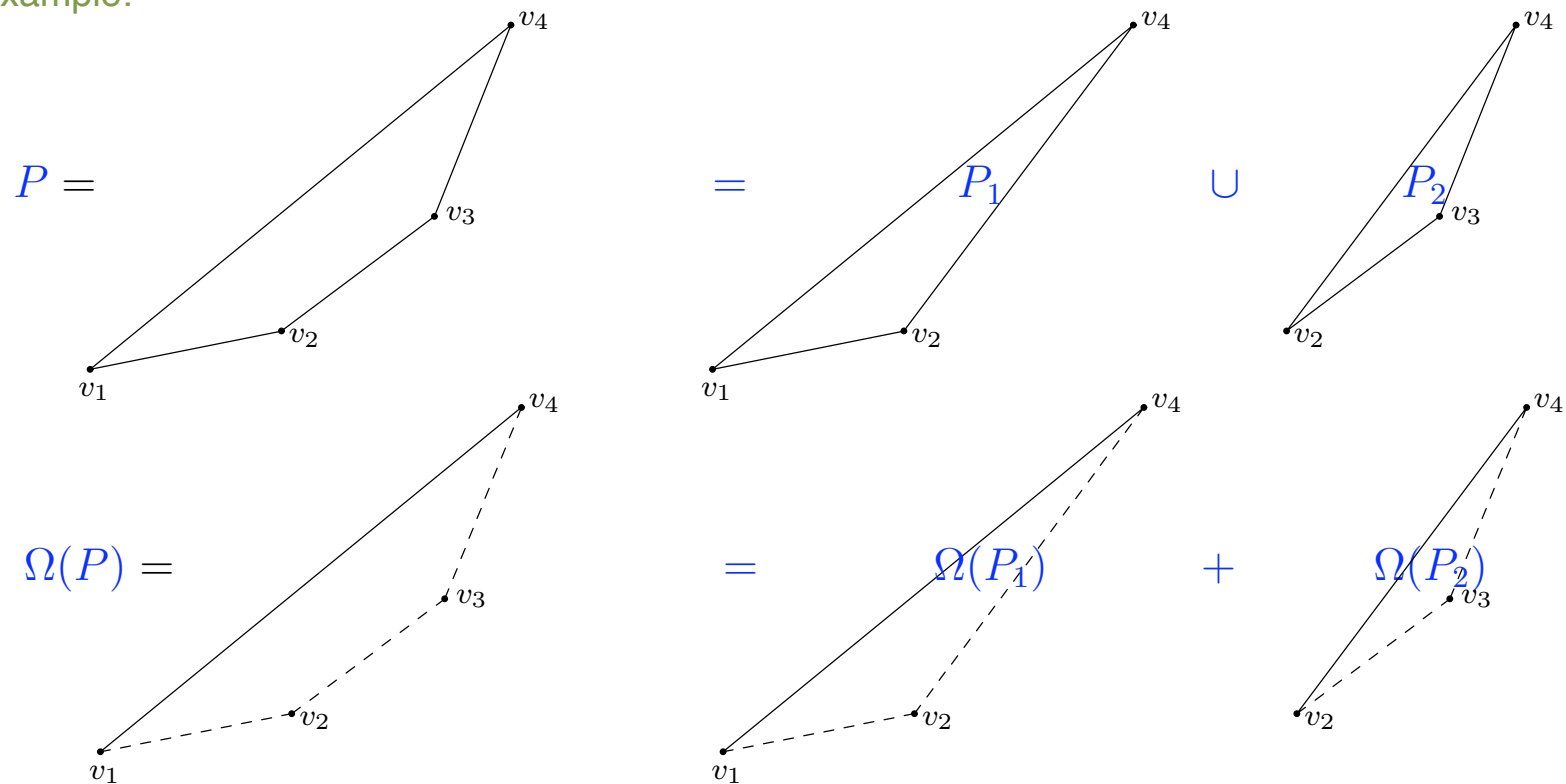
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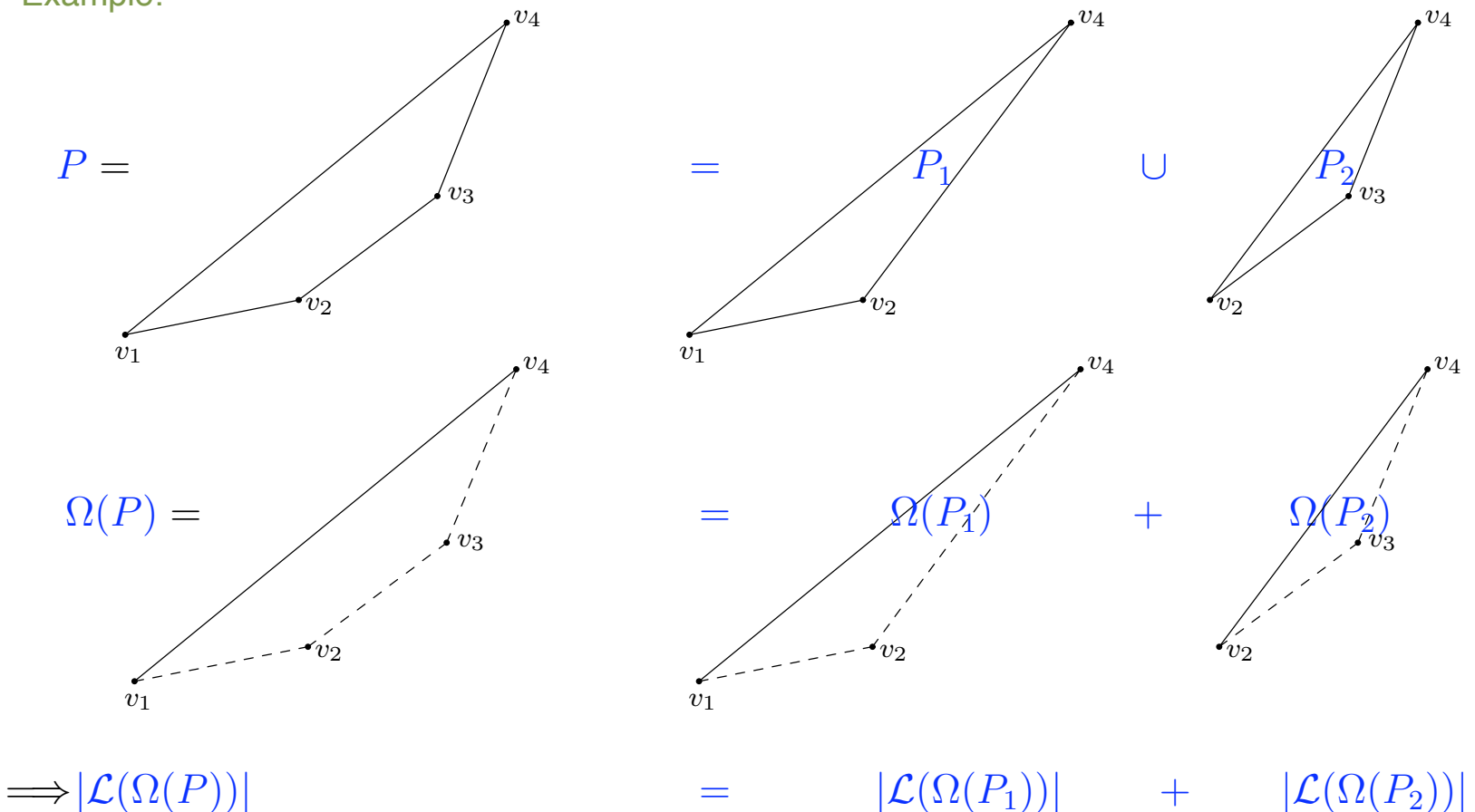
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For any triangulation (without introducing new vertices) $P_1 \cup \dots \cup P_k$ of a lattice-face polytope P , (note that (iv) implies that all P_i are lattice-face polytopes,) we have that

$$\Omega(P) = \bigoplus_{i=1}^k \Omega(P_i), \text{ which implies that } |\mathcal{L}(\Omega(P))| = \sum_{i=1}^k |\mathcal{L}(\Omega(P_i))|.$$

Example:



However, for any triangulation (without introducing new vertices) $P_1 \cup \cdots \cup P_k$, we have that

$$\text{Vol}(P) = \sum_{i=1}^k \text{Vol}(P_i).$$

Comparing this with

$$|\mathcal{L}(\Omega(P))| = \sum_{i=1}^k |\mathcal{L}(\Omega(P_i))|,$$

we conclude that, to prove Theorem 9 ($\text{Vol}(P) = |\mathcal{L}(\Omega(P))|$), it is enough to prove the the case when P is a lattice-face d -simplex, i.e., P has $d + 1$ vertices which are affinely independent.

Idea of the Proof

We will use two dimensional lattice-face simplices to illustrate the idea of our proof.

Assume P is a 2-dimensional lattice-face simplex with vertex set $V = \{v_1, v_2, v_3\}$, where the coordinates of v_i are $\begin{pmatrix} x_i \\ y_i \end{pmatrix}$.

WLOG, we assume that v_1, v_2, v_3 are in an order such that both $\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$ and $\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}$ are positive. In other words, v_1, v_2, v_3 are in counterclockwise order and $x_1 < x_2$.

We can define an affine transformation T which maps $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \left(\begin{array}{c} x \\ \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}} \end{array} \right)$. One can

check that

1. T gives a bijection between the lattice points in $\Omega(P)$ and the lattice points in $\Omega(T(P))$. Therefore, we want to show that

$$\text{Vol}(P) = |\mathcal{L}(\Omega(T(P)))|.$$

2. $T(P)$ is a lattice-face polytope, as well.

Let $P' = T(P)$. Then its vertex set is $V' = \{v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}\},$

$$\text{where } y'_3 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}}.$$

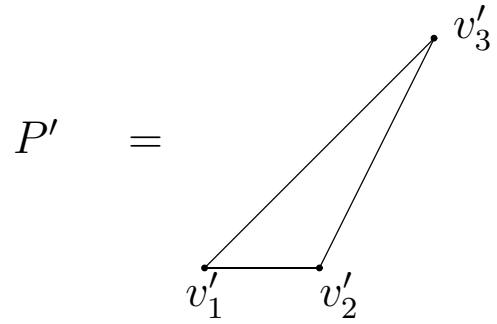
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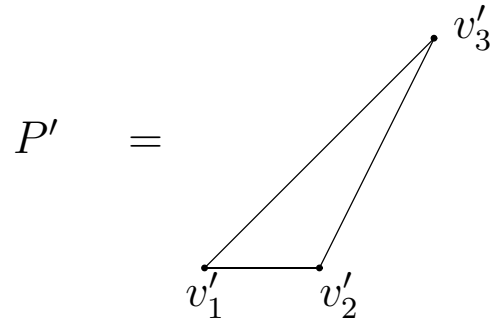
By our assumption, $y'_3 > 0$. There are 3 cases for the position of the vertices of P' :

(i) $x_1 < x_2 < x_3$; (ii) $x_1 < x_3 < x_2$; (iii) $x_3 < x_1 < x_2$.

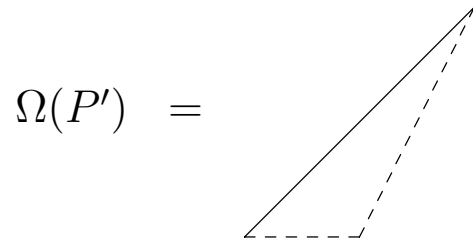
(i) $x_1 < x_2 < x_3$:



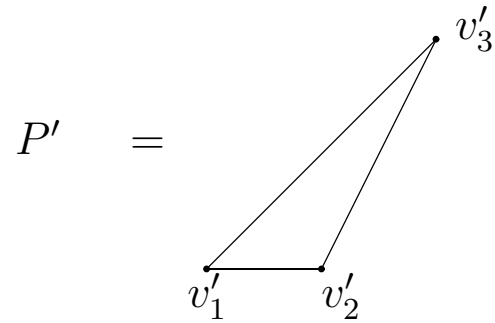
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

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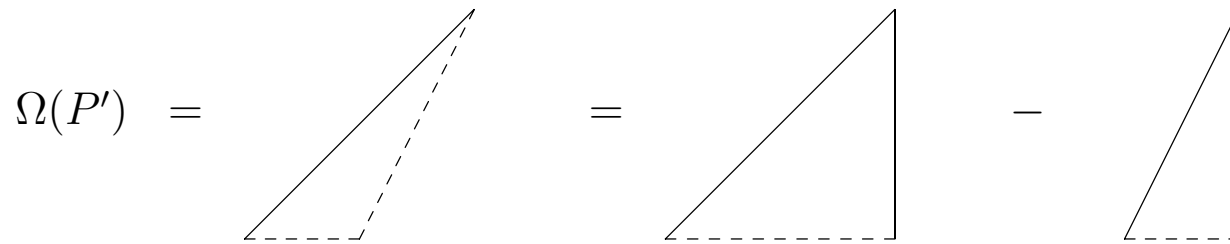
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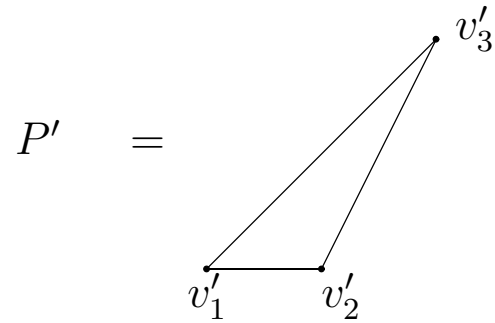
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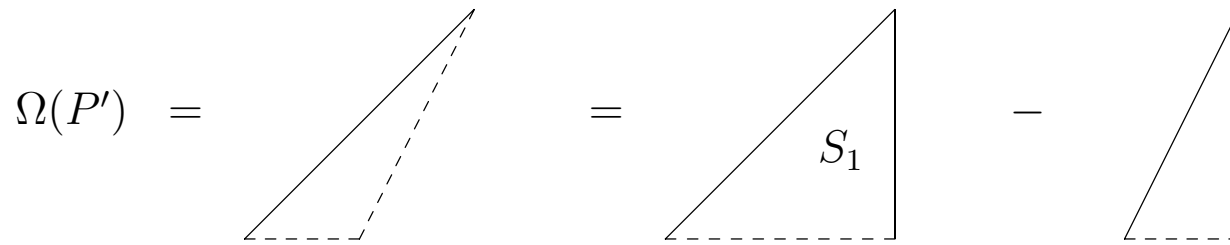
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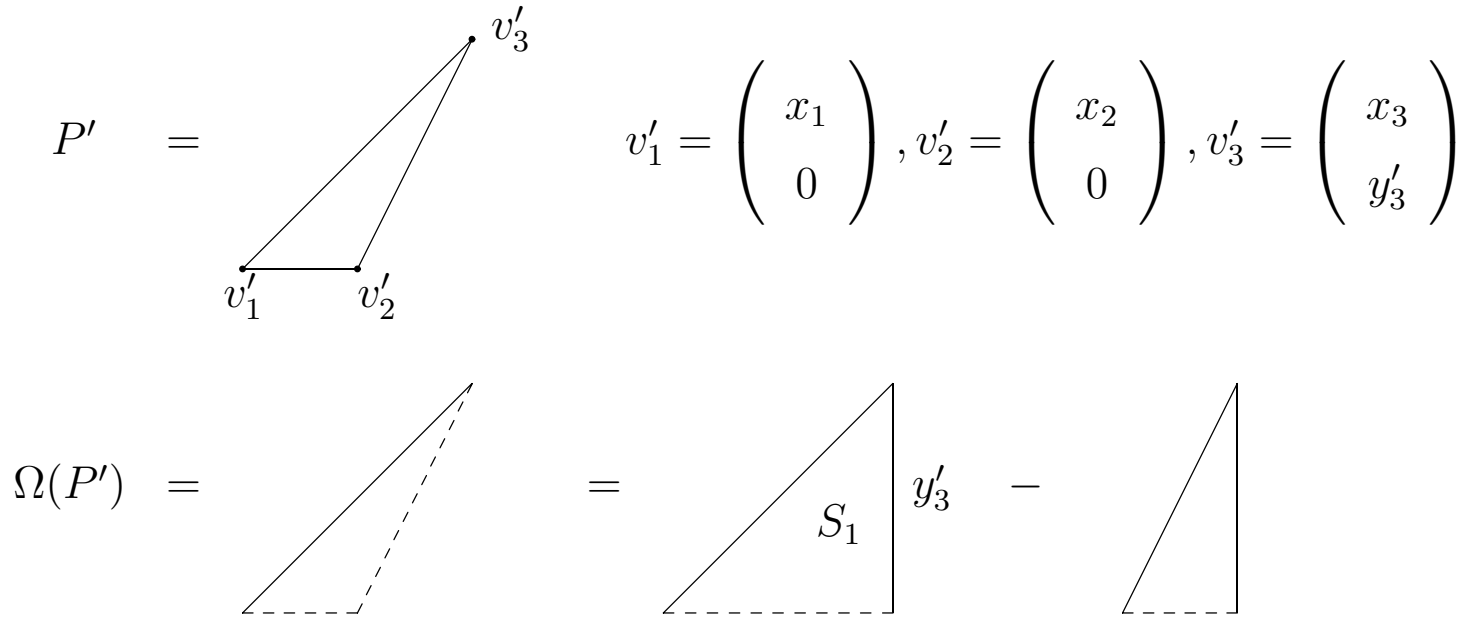
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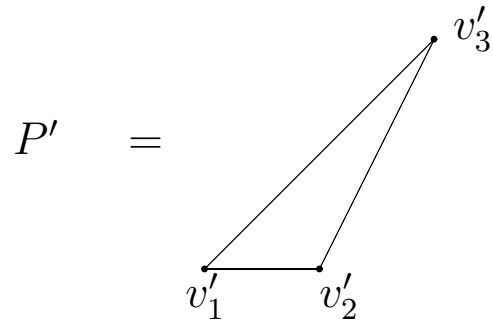
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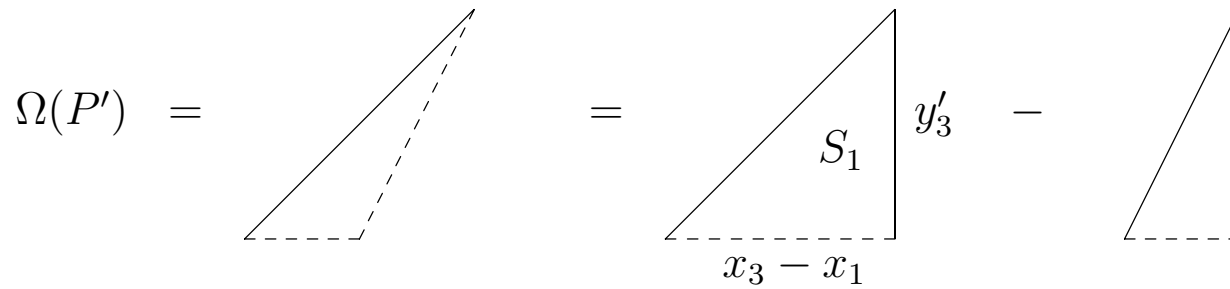
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$$\Omega(P') = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} S_1 y'_3 - \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}$$

$$\begin{array}{c} x_3 - x_1 \\ \parallel \\ \left| \begin{array}{cc} 1 & x_1 \\ 1 & x_3 \end{array} \right| \end{array}$$

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$$\begin{array}{c} x_3 - x_1 \\ \parallel \\ \left| \begin{array}{cc} 1 & x_1 \\ 1 & x_3 \end{array} \right| \end{array} \qquad \begin{array}{c} x_3 - x_2 \\ \parallel \\ \left| \begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \right| \end{array}$$

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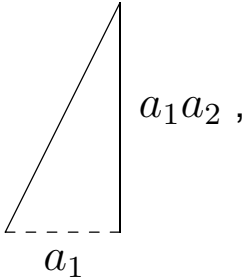
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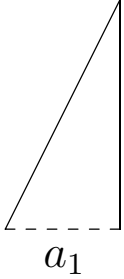
$$\begin{array}{c} x_3 - x_1 \\ \left| \begin{array}{c} \parallel \\ 1 \quad x_1 \\ 1 \quad x_3 \end{array} \right| \end{array} \qquad \begin{array}{c} x_3 - x_2 \\ \left| \begin{array}{c} \parallel \\ 1 \quad x_2 \\ 1 \quad x_3 \end{array} \right| \end{array}$$

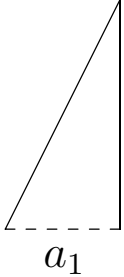
Because P' is a lattice-face polytope, it is not hard to show that y'_3 is a multiple of both $\left| \begin{array}{c} 1 \quad x_1 \\ 1 \quad x_3 \end{array} \right|$

and $\left| \begin{array}{c} 1 \quad x_2 \\ 1 \quad x_3 \end{array} \right|$.

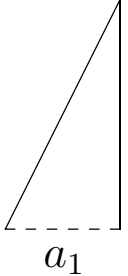
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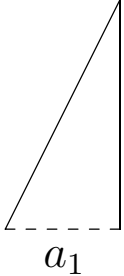
For any positive integers a_1, a_2 , if $S =$  $a_1 a_2$, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$.

Therefore, we define $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$, for any positive integers a_1, a_2 .

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$$|\mathcal{L}(S_1)| = \sum_{s_1=1} \left| \begin{array}{c|c} 1 & x_1 \\ \hline 1 & x_3 \end{array} \right| \left(\left(y_3' / \left| \begin{array}{c|c} 1 & x_1 \\ \hline 1 & x_3 \end{array} \right| \right)_{s_1} \right) = f_2 \left(\left(\begin{array}{c|c} 1 & x_1 \\ \hline 1 & x_3 \end{array} \right), y_3' / \left(\begin{array}{c|c} 1 & x_1 \\ \hline 1 & x_3 \end{array} \right) \right), \text{ and}$$

For any positive integers a_1, a_2 , if $S =$  $a_1 a_2$, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$.

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$$|\mathcal{L}(S_1)| = \sum_{s_1=1} \left| \begin{array}{cc} 1 & x_1 \\ 1 & x_3 \end{array} \right| \left(\left| \begin{array}{cc} y'_3/ & 1 \\ 1 & x_3 \end{array} \right| \right)^{s_1} = f_2 \left(\left| \begin{array}{cc} 1 & x_1 \\ 1 & x_3 \end{array} \right|, y'_3/ \left| \begin{array}{cc} 1 & x_1 \\ 1 & x_3 \end{array} \right| \right), \text{ and}$$

$$|\mathcal{L}(S_2)| = \sum_{s_1=1} \left| \begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \right| \left(\left| \begin{array}{cc} y'_3/ & 1 \\ 1 & x_3 \end{array} \right| \right)^{s_1} = f_2 \left(\left| \begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \right|, y'_3/ \left| \begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \right| \right).$$

For any positive integers a_1, a_2 , if $S = \begin{array}{c} \triangle \\ \hline a_1 \end{array} a_1 a_2$, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$.

Therefore, we define $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$, for any positive integers a_1, a_2 . Thus,

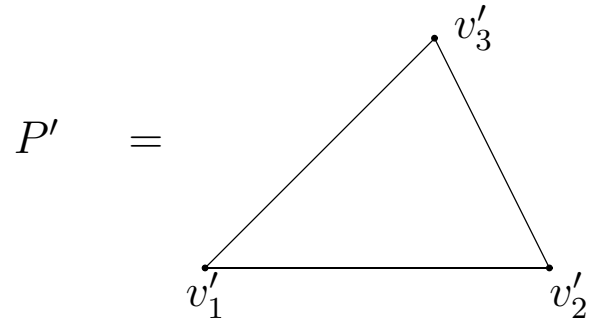
$$|\mathcal{L}(S_1)| = \sum_{s_1=1} \left| \begin{array}{cc|c} 1 & x_1 & \\ 1 & x_3 & \end{array} \right| \left(\begin{array}{cc|c} y'_3/ & 1 & x_1 \\ & 1 & x_3 \end{array} \right)_{s_1} = f_2 \left(\left(\begin{array}{cc|c} 1 & x_1 & \\ 1 & x_3 & \end{array} \right), y'_3/ \left(\begin{array}{cc|c} 1 & x_1 & \\ 1 & x_3 & \end{array} \right) \right), \text{ and}$$

$$|\mathcal{L}(S_2)| = \sum_{s_1=1} \left| \begin{array}{cc|c} 1 & x_2 & \\ 1 & x_3 & \end{array} \right| \left(\begin{array}{cc|c} y'_3/ & 1 & x_2 \\ & 1 & x_3 \end{array} \right)_{s_1} = f_2 \left(\left(\begin{array}{cc|c} 1 & x_2 & \\ 1 & x_3 & \end{array} \right), y'_3/ \left(\begin{array}{cc|c} 1 & x_2 & \\ 1 & x_3 & \end{array} \right) \right).$$

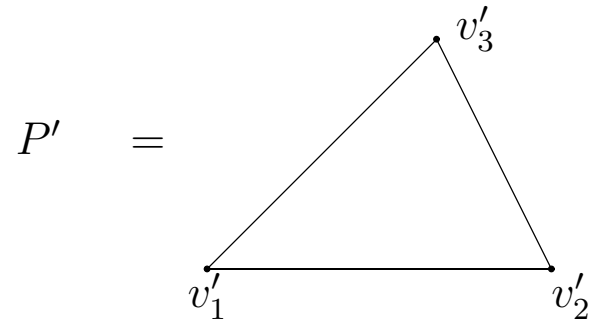
Hence, for case (i) $x_1 < x_2 < x_3$: $|\mathcal{L}(\Omega(P))| = |\mathcal{L}(\Omega(P'))| = |\mathcal{L}(S_1)| - |\mathcal{L}(S_2)|$

$$= f_2 \left(\left(\begin{array}{cc|c} 1 & x_1 & \\ 1 & x_3 & \end{array} \right), y'_3/ \left(\begin{array}{cc|c} 1 & x_1 & \\ 1 & x_3 & \end{array} \right) \right) - f_2 \left(\left(\begin{array}{cc|c} 1 & x_2 & \\ 1 & x_3 & \end{array} \right), y'_3/ \left(\begin{array}{cc|c} 1 & x_2 & \\ 1 & x_3 & \end{array} \right) \right).$$

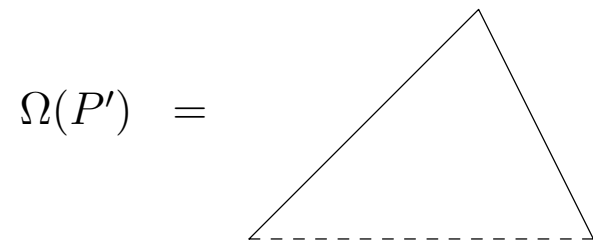
(ii) $x_1 < x_3 < x_2$:



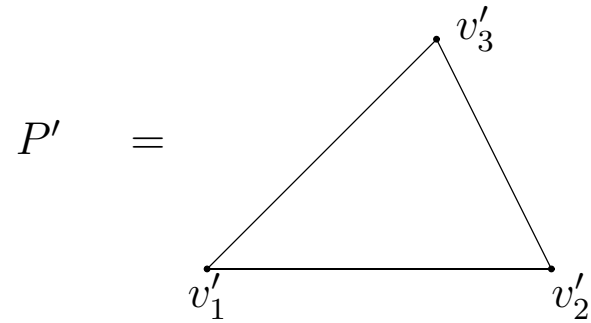
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

(ii) $x_1 < x_3 < x_2$:

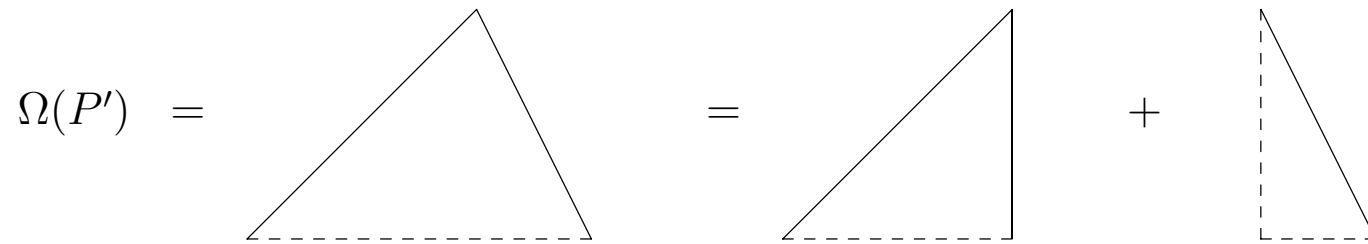
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$



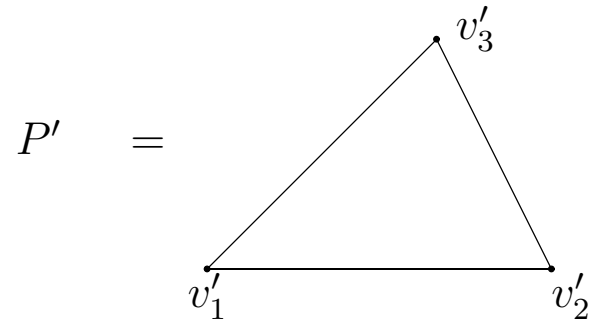
(ii) $x_1 < x_3 < x_2$:



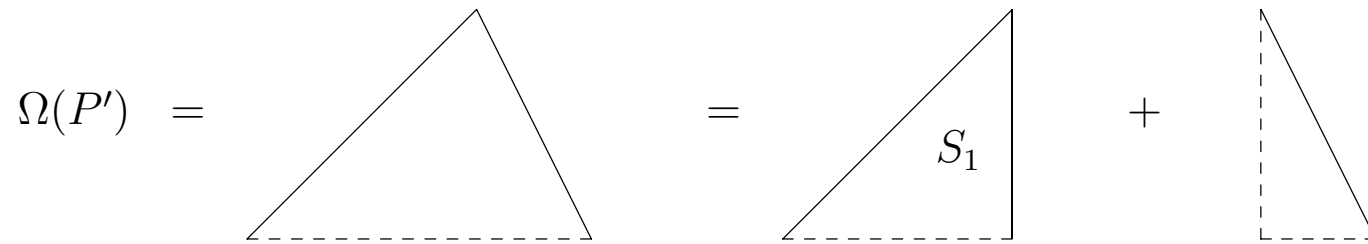
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$



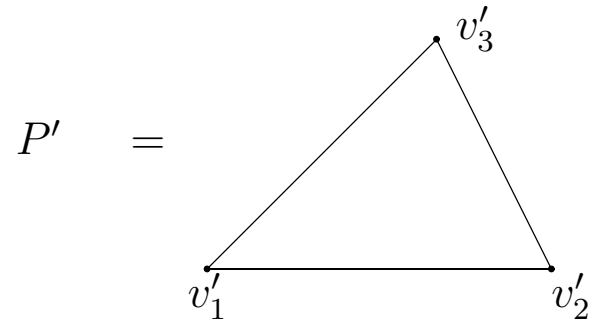
(ii) $x_1 < x_3 < x_2$:



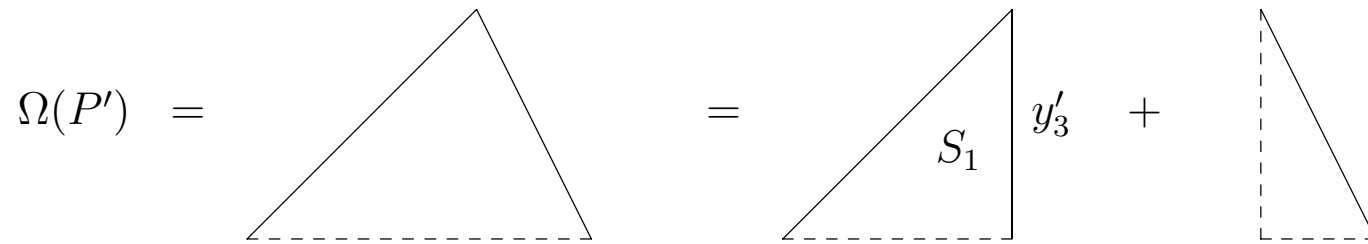
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$



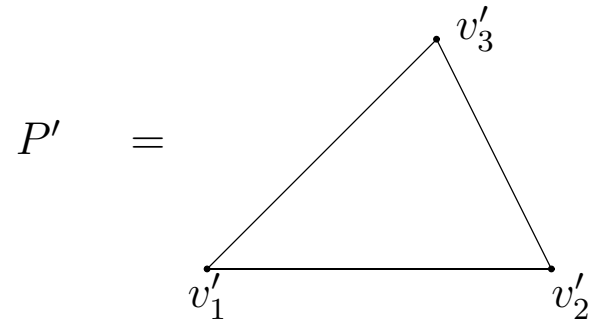
(ii) $x_1 < x_3 < x_2$:



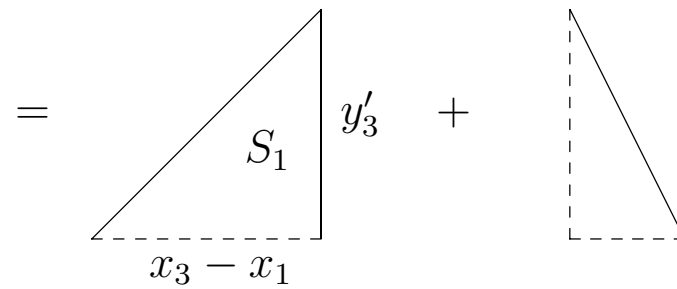
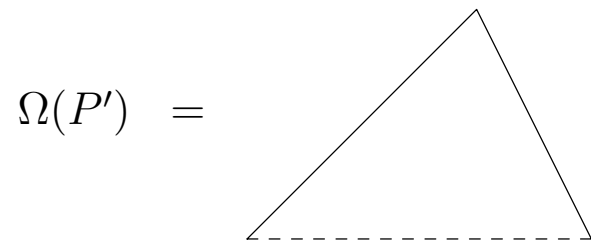
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$



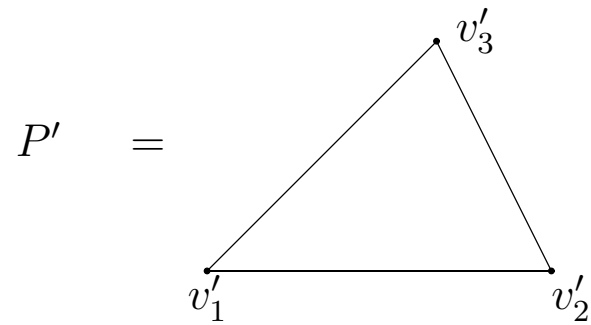
(ii) $x_1 < x_3 < x_2$:



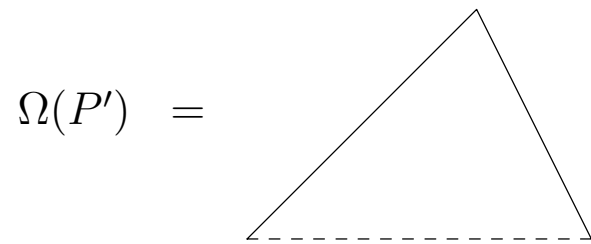
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$



(ii) $x_1 < x_3 < x_2$:

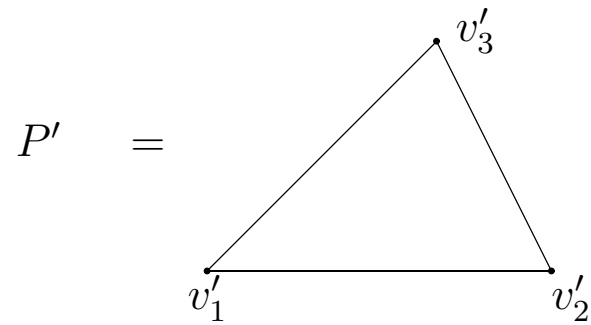


$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

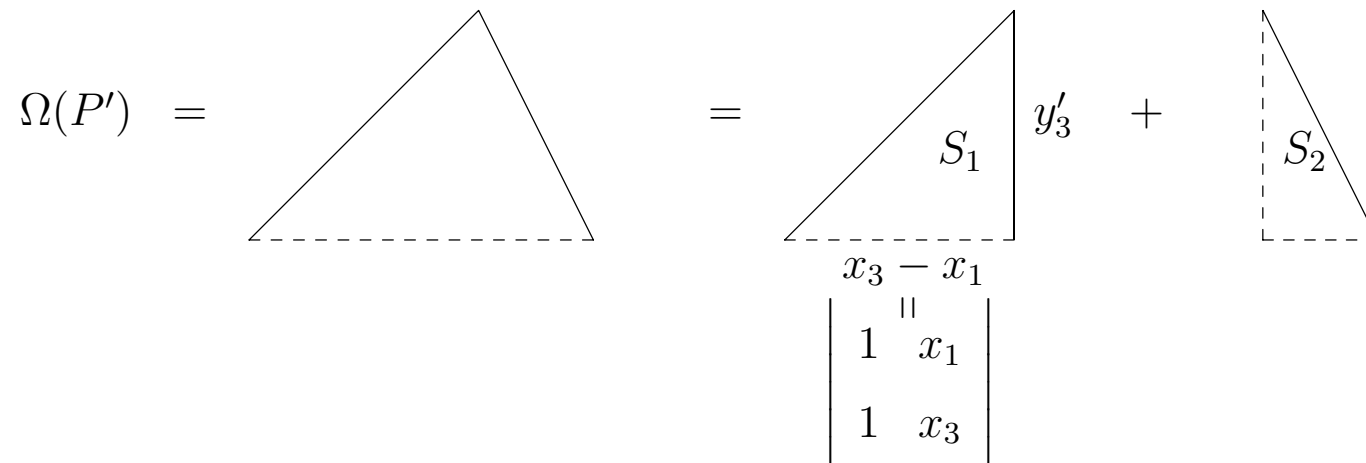


=

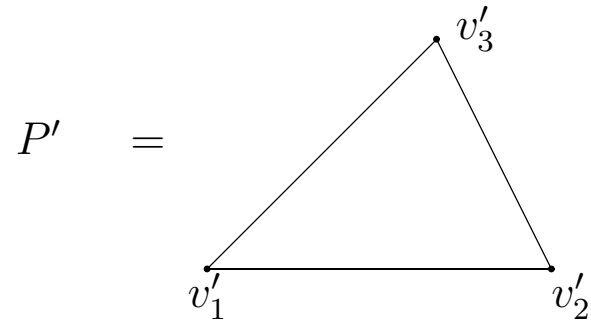
(ii) $x_1 < x_3 < x_2$:



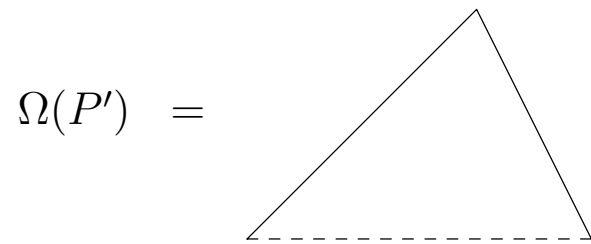
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$



(ii) $x_1 < x_3 < x_2$:



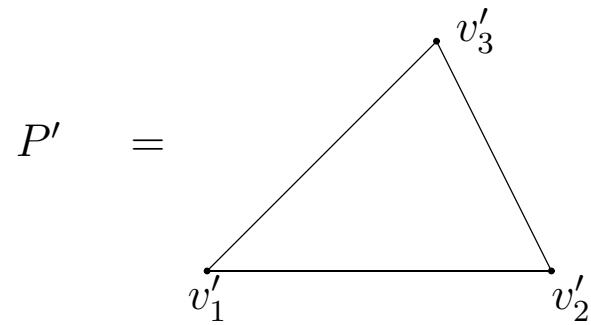
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$



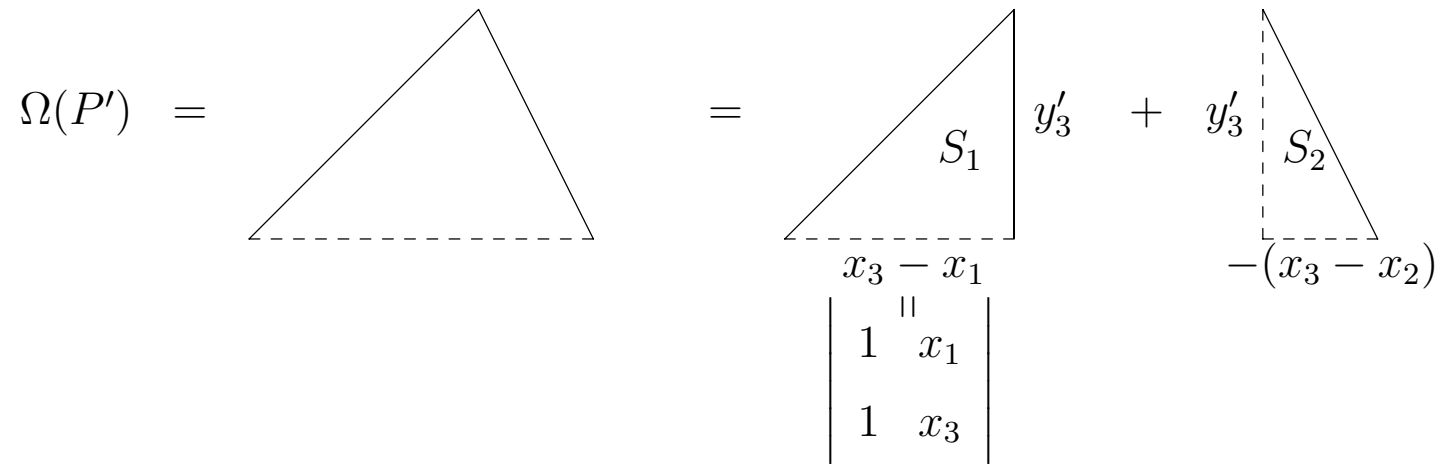
=

$$\begin{matrix} x_3 - x_1 \\ \parallel \\ \left| \begin{matrix} 1 & x_1 \\ 1 & x_3 \end{matrix} \right| \end{matrix}$$

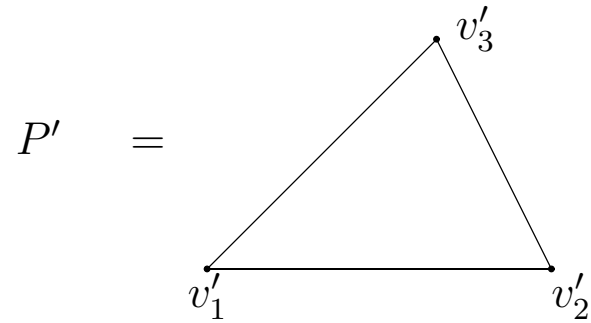
(ii) $x_1 < x_3 < x_2$:



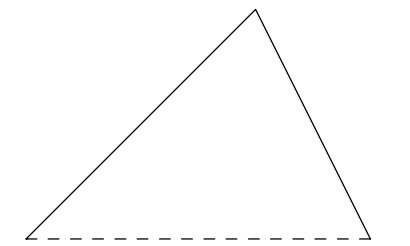
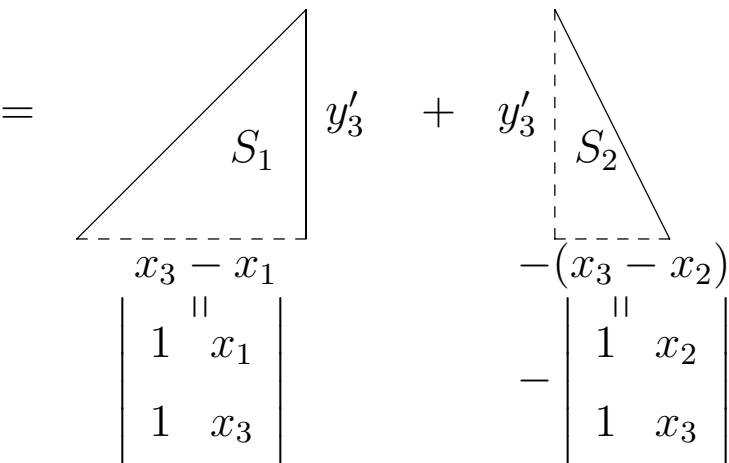
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$



(ii) $x_1 < x_3 < x_2$:

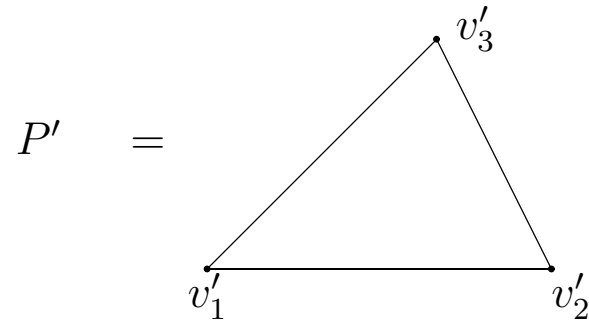


$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

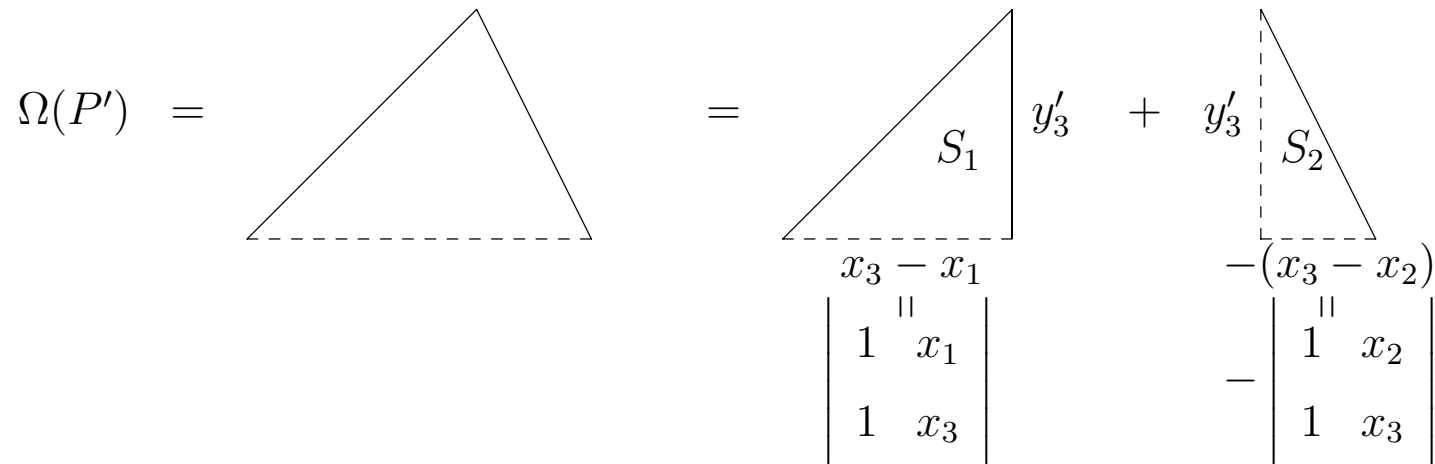
$\Omega(P')$ =  = 

$$= \begin{matrix} x_3 - x_1 \\ \parallel \\ \left| \begin{matrix} 1 & x_1 \\ 1 & x_3 \end{matrix} \right| \end{matrix} y'_3 + \begin{matrix} y'_3 \\ \parallel \\ - \left(x_3 - x_2 \right) \\ \parallel \\ - \left| \begin{matrix} 1 & x_2 \\ 1 & x_3 \end{matrix} \right| \end{matrix}$$

(ii) $x_1 < x_3 < x_2$:

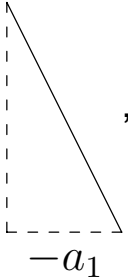


$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

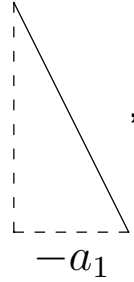


Clearly $|\mathcal{L}(S_1)| = f_2 \left(\left(\begin{array}{c|c} 1 & x_1 \\ \hline 1 & x_3 \end{array} \right), y'_3/ \left(\begin{array}{c|c} 1 & x_1 \\ \hline 1 & x_3 \end{array} \right) \right)$, but what is $|\mathcal{L}(S_2)|$?

For any negative integers a_1, a_2 , if $S = a_1 a_2$,



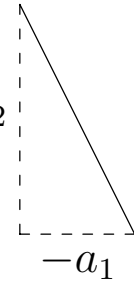
For any negative integers a_1, a_2 , if $S = \begin{matrix} a_1 a_2 \\ \vdots \\ -a_1 \end{matrix}$, then $|\mathcal{L}(S)| = \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1$



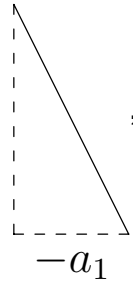
For any negative integers a_1, a_2 , if $S = \begin{matrix} a_1 a_2 \\ \vdots \\ -a_1 \end{matrix}$, then

$$\begin{aligned} |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\ &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \end{aligned}$$

For any negative integers a_1, a_2 , if $S = \begin{matrix} a_1 a_2 \\ \vdots \\ -a_1 \end{matrix}$, ,

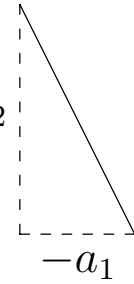


$$\begin{aligned} \text{then } |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\ &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \\ &= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1) \end{aligned}$$

For any negative integers a_1, a_2 , if $S = a_1 a_2$ , then

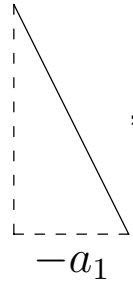
$$\begin{aligned}
 |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\
 &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \\
 &= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1) \\
 &= -a_2 \frac{a_1}{2} (a_1 + 1)
 \end{aligned}$$

For any negative integers a_1, a_2 , if $S = \begin{matrix} a_1 a_2 \\ \vdots \\ -a_1 \end{matrix}$, then



$$\begin{aligned} |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\ &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \\ &= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1) \\ &= -a_2 \frac{a_1}{2} (a_1 + 1) \end{aligned}$$

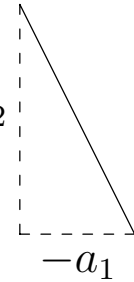
Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1$

For any negative integers a_1, a_2 , if $S = \begin{matrix} a_1 a_2 \\ \vdots \\ -a_1 \end{matrix}$, 

$$\begin{aligned} \text{then } |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\ &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \\ &= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1) \\ &= -a_2 \frac{a_1}{2} (a_1 + 1) \end{aligned}$$

Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 = \sum_{s_1=1}^{a_1} a_2 s_1$

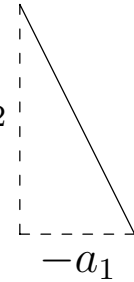
For any negative integers a_1, a_2 , if $S = \begin{matrix} a_1 a_2 \\ \vdots \\ -a_1 \end{matrix}$, then



$$\begin{aligned} |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\ &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \\ &= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1) \\ &= -a_2 \frac{a_1}{2} (a_1 + 1) \end{aligned}$$

Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 = \sum_{s_1=1}^{a_1} a_2 s_1 = a_2 \frac{a_1}{2} (a_1 + 1)$.

For any negative integers a_1, a_2 , if $S = \begin{matrix} a_1 a_2 \\ \vdots \\ -a_1 \end{matrix}$, then

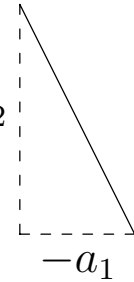


$$\begin{aligned} |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\ &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \\ &= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1) \\ &= -a_2 \frac{a_1}{2} (a_1 + 1) \end{aligned}$$

Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 = \sum_{s_1=1}^{a_1} a_2 s_1 = a_2 \frac{a_1}{2} (a_1 + 1)$.

Because $a_2 \frac{a_1}{2} (a_1 + 1)$ is a polynomial in a_1, a_2 , we can extend the domain of f_2 from \mathbb{N}^2 to \mathbb{Z}^2 or even \mathbb{R}^2 .

For any negative integers a_1, a_2 , if $S = \begin{matrix} a_1 a_2 \\ \vdots \\ -a_1 \end{matrix}$, then



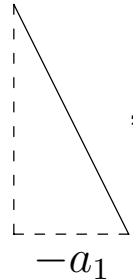
$$\begin{aligned} |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\ &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \\ &= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1) \\ &= -a_2 \frac{a_1}{2} (a_1 + 1) \end{aligned}$$

Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 = \sum_{s_1=1}^{a_1} a_2 s_1 = a_2 \frac{a_1}{2} (a_1 + 1)$.

Because $a_2 \frac{a_1}{2} (a_1 + 1)$ is a polynomial in a_1, a_2 , we can extend the domain of f_2 from \mathbb{N}^2 to \mathbb{Z}^2 or even \mathbb{R}^2 .

Then

$$|\mathcal{L}(S)| = -f_2(a_1, a_2).$$

For any negative integers a_1, a_2 , if $S = a_1 a_2$ , then

$$\begin{aligned}
 |\mathcal{L}(S)| &= \sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} 1 \\
 &= \sum_{s_1=1}^{-a_1-1} (-a_2 s_1) \\
 &= -a_2 \frac{1}{2} (-a_1 - 1)(-a_1) \\
 &= -a_2 \frac{a_1}{2} (a_1 + 1)
 \end{aligned}$$

Recall that for any $a_1, a_2 \in \mathbb{N}$, $f_2(a_1, a_2) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} 1 = \sum_{s_1=1}^{a_1} a_2 s_1 = a_2 \frac{a_1}{2} (a_1 + 1)$.

Because $a_2 \frac{a_1}{2} (a_1 + 1)$ is a polynomial in a_1, a_2 , we can extend the domain of f_2 from \mathbb{N}^2 to \mathbb{Z}^2 or even \mathbb{R}^2 .

Then

$$|\mathcal{L}(S)| = -f_2(a_1, a_2).$$

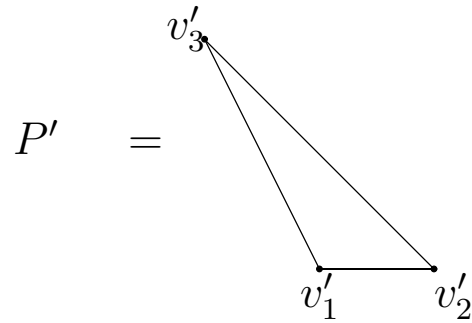
Thus,

$$|\mathcal{L}(S_2)| = \sum_{s_1=1}^{-\left| \begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \right|^{-1} \left(-y'_3 / \begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \right)_{s_1}} 1 = -f_2 \left(\begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \middle| , y'_3 / \begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \right).$$

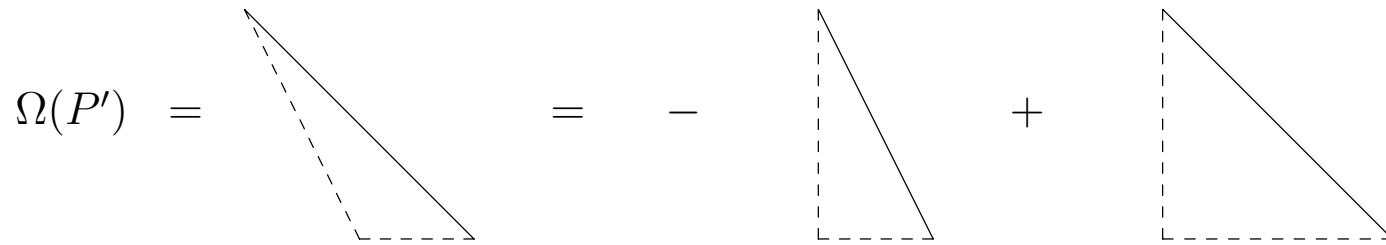
Hence, for case (ii) $x_1 < x_3 < x_2$: $|\mathcal{L}(\Omega(P))| = |\mathcal{L}(\Omega(P'))| = |\mathcal{L}(S_1)| + |\mathcal{L}(S_2)| =$

$$f_2 \left(\begin{array}{cc} 1 & x_1 \\ 1 & x_3 \end{array} \middle| , y'_3 / \begin{array}{cc} 1 & x_1 \\ 1 & x_3 \end{array} \right) - f_2 \left(\begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \middle| , y'_3 / \begin{array}{cc} 1 & x_2 \\ 1 & x_3 \end{array} \right).$$

(iii) $x_3 < x_1 < x_2$:



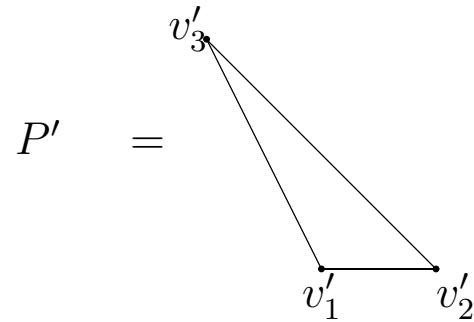
$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$



(iii) $x_3 < x_1 < x_2$:

$$\begin{aligned}
 P' &= \begin{array}{c} v'_3 \\ \diagdown \quad \diagup \\ v'_1 \quad v'_2 \end{array} & v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix} \\
 \\ \\
 \Omega(P') &= \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} &= & - y'_3 \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ S_1 \end{array} & + & y'_3 \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ S_2 \end{array} \\
 & & & - \begin{pmatrix} x_3 - x_1 \\ \parallel \\ 1 \quad x_1 \\ 1 \quad x_3 \end{pmatrix} & & - \begin{pmatrix} x_3 - x_2 \\ \parallel \\ 1 \quad x_2 \\ 1 \quad x_3 \end{pmatrix}
 \end{aligned}$$

(iii) $x_3 < x_1 < x_2$:



$$v'_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, v'_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} x_3 \\ y'_3 \end{pmatrix}$$

$$\Omega(P') = \text{[Diagram of a triangle with dashed lines]} = - y'_3 \begin{matrix} \text{[Diagram of triangle } S_1 \text{ with dashed lines]} \\ - \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \end{matrix} + y'_3 \begin{matrix} \text{[Diagram of triangle } S_2 \text{ with dashed lines]} \\ - \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} \end{matrix}$$

As before, we have that $|\mathcal{L}(\Omega(P))| = |\mathcal{L}(\Omega(P'))| = -|\mathcal{L}(S_1)| + |\mathcal{L}(S_2)|$

$$= f_2 \left(\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix}, y'_3/ \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \right) - f_2 \left(\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix}, y'_3/ \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} \right).$$

Therefore, for any of the three cases,

$$|\mathcal{L}(\Omega(P))| = f_2 \left(\left| \begin{array}{cc|c} 1 & x_1 & \\ \hline 1 & x_3 & \end{array} \right|, y'_3 / \left| \begin{array}{cc|c} 1 & x_1 & \\ \hline 1 & x_3 & \end{array} \right| \right) - f_2 \left(\left| \begin{array}{cc|c} 1 & x_2 & \\ \hline 1 & x_3 & \end{array} \right|, y'_3 / \left| \begin{array}{cc|c} 1 & x_2 & \\ \hline 1 & x_3 & \end{array} \right| \right),$$

$$\text{where } y'_3 = \frac{\left| \begin{array}{ccc|c} 1 & x_1 & y_1 & \\ \hline 1 & x_2 & y_2 & \\ 1 & x_3 & y_3 & \end{array} \right|}{\left| \begin{array}{cc|c} 1 & x_1 & \\ \hline 1 & x_2 & \end{array} \right|}.$$

Therefore, for any of the three cases,

$$|\mathcal{L}(\Omega(P))| = f_2 \left(\left| \begin{array}{cc|c} 1 & x_1 & y_3' \\ 1 & x_3 & \end{array} \right|, y_3' / \left| \begin{array}{cc|c} 1 & x_1 & \\ 1 & x_3 & \end{array} \right| \right) - f_2 \left(\left| \begin{array}{cc|c} 1 & x_2 & y_3' \\ 1 & x_3 & \end{array} \right|, y_3' / \left| \begin{array}{cc|c} 1 & x_2 & \\ 1 & x_3 & \end{array} \right| \right),$$

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Recall that $f_2(a_1, a_2) = a_2 \frac{a_1}{2} (a_1 + 1)$, we can calculate that

$$|\mathcal{L}(\Omega(P))| = \frac{1}{2} \left| \begin{array}{ccc|c} 1 & x_1 & y_1 & \\ 1 & x_2 & y_2 & \\ 1 & x_3 & y_3 & \end{array} \right| = \text{Vol}(P).$$

Therefore, for any of the three cases,

$$|\mathcal{L}(\Omega(P))| = f_2 \left(\begin{array}{c|c} 1 & x_1 \\ 1 & x_3 \end{array} \middle|, y'_3 / \begin{array}{c|c} 1 & x_1 \\ 1 & x_3 \end{array} \right) - f_2 \left(\begin{array}{c|c} 1 & x_2 \\ 1 & x_3 \end{array} \middle|, y'_3 / \begin{array}{c|c} 1 & x_2 \\ 1 & x_3 \end{array} \right),$$

$$\text{where } y'_3 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}}.$$

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This completes the proof of Theorem 9 for dimension 2.

Further Discussion

We have an alternative definition of lattice-face polytopes, which is equivalent to the original definition we gave earlier. Indeed, a d -polytope on a vertex set V is a lattice-face polytope if and only if for all $k : 0 \leq k \leq d - 1$,

$$(\star) \quad \text{for any } (k + 1)\text{-subset } U \subset V, \\ \pi^{d-k}(\mathcal{L}(H_U)) = \mathbb{Z}^k,$$

where H_U is the affine space spanned by U . In other words, after dropping the last $d - k$ coordinates of the lattice of H_U , we get the k -dimensional lattice.

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Therefore, one may ask

Question: If P is a polytope that satisfies (\star) for all $k \in K$, where K is a fixed subset of $\{0, 1, \dots, d-1\}$, can we say something about the Ehrhart polynomial of P ?

A conjecture

A special set K can be chosen as the set of consecutive integers from 0 to d' , where d' is an integer no greater than $d - 1$. Based on some examples in this case, the Ehrhart polynomials seems to follow a certain pattern, so we conjecture the following:

Conjecture 10. *Given $d' \leq d - 1$, if P is a d -polytope with vertex set V such that $\forall k : 0 \leq k \leq d'$, (\star) is satisfied, then for $0 \leq k \leq d'$, the coefficient of m^k in $i(P, m)$ is the same as in $i(\pi^{d-d'}(P), m)$. In other words,*

$$i(P, m) = i(\pi^{d-d'}(P), m) + \sum_{i=d'+1}^d c_i m^i.$$

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Example: $P = \text{conv}\{(0, 0, 0), (4, 0, 0), (3, 6, 0), (2, 2, 2)\}$. One can check that P satisfies (\star) for $k = 0, 1$ but not for $k = 2$.

$$i(P, m) = 8m^3 + 10m^2 + 4m + 1,$$

where $4m + 1$ is the Ehrhart polynomial of $\pi^2(P) = [0, 4]$.