The octahedron recurrence and combinatorics of arrays

V. I. Danilov and G.A. Koshevoy *
CEMI RAS, Nahimovskii pr. 47, 117418 Moscow, and
Poncelet Lab. (UMI 2615 of CNRS and Independent University of Moscow)
e-mail: vdanilov43@mail.ru, koshevoy@cemi.rssi.ru

1 Introduction

In [1] it was proposed a functional approach to combinatorics of Young tableaux. This approach proved itself extremely useful for combinatorics of Littlewood-Richardson coefficients and solutions to the Horn problem, see [2, 3, 8, 11]. Due to this line of approach, LR-coefficients count integer-valued discrete concave functions on a triangle grid in $\mathbb{Z}^2$ with prescribed boundary values, and the Hermitian matrices A, B and $A+B$ with spectra α, β and γ, respectively, exist iff there exists a discrete concave function on a triangular grid with the increments α, β and γ along the sides of the triangle ([3, 11]).

An interesting bijection between two sets constituted from some pairs of discrete concave functions was constructed in [12]. This bijection is based upon the octahedron recurrence, or the tropicalization of the discrete Hirota equation. Specifically the octahedron recurrence gradually propagates a pair of discrete concave functions at two adjoint faces (grids) of a tetrahedron to a pair of functions on other two faces. In [6] we have shown that the resulting functions are indeed discrete concave, and this is because of a relation between the octahedron recurrence and discrete concavity in dimension 3.

Here we show that the modified RSK for arrays[1] can be obtained by the octahedral recurrence. Specifically, to an array we associate a supermodular function on $\mathbb{Z}^2_+$. We locate this function at a rectangular modular face of a prism, and we set the null function at the non-modular face, and again the null function at the "bottom" (with respect to the propagation vector) triangle face. These initial data we propagate using the octahedron recurrence to

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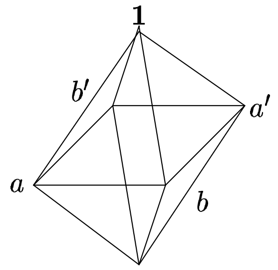
[1] For integer-valued arrays, the modified (crystal) RSK differs from the the RSK by changing Q-symbol by the Schutzenberger involution of it. This slight modification is forced by tensor product on crystals (see [4, 5]).

the whole prism, and get a polarized function on the prism (discrete concave by modulo adding two one-dimensional concave functions). Restrictions of this function to two other modular faces are so-called vertically strip concave (VS-concave) and horizontally strip concave functions (HS-concave), respectively. This construction is invertible (the octahedron recurrence with the opposite propagation vector). Thus we get a bijection, the *functional RSK*, between the set of arrays and pairs of VS-concave and HS-concave functions of equal "shape". The set of integer-valued VS-concave (HS-concave) functions is in bijection with the set of semi-standard Young tableaux [4]. Thus, for the integer-valued setup, this construction provides a bijection between arrays and pairs of semi-standard Young tableaux of the same shape. This, of course, reminds the RSK correspondence, but indeed this bijection is the modified RSK. Note, that the Schutzenberger involution also might be obtained by the octahedron recurrence. Let us remark an advantage of the functional form of RSK: the direct and inverse bijections are done symmetrically.

Finally, the complexity of the functional RSK (direct and reverse) is $O(\max(n,m)^3)$, and the usual RSK algorithm has complexity $O((\sum_{ij}(a(i,j))^{3/2})$.

2 Octahedron recurrence

The main idea of the octahedron recurrence is rather transparent. Specifically, consider the octahedron



Picture 1.

with the vertexes $\mathbf{0}, a, a', b, b'$ and $\mathbf{1}$. Let f be a real-valued function given at the points $\mathbf{0}, a, a', b, b'$. Then we can *propagate* f to the point $\mathbf{1}$ by the following rule

$$f(\mathbf{1}) = \max(f(a) + f(a'), f(b) + f(b')) - f(\mathbf{0}).$$

We refer to [13] for justifications of this rule and interesting examples of appearances of this rule in combinatorics. Rather unexpectedly this related to flips in [7]. We want to point out a relation of this rule to concavity.

Specifically, suppose $f(b) + f(b') = \max(f(a) + f(a'), f(b) + f(b'))$. Then, we have $f(\mathbf{0}) + f(\mathbf{1}) = f(b) + f(b')$. This means that the restriction of the function to the rhombus $\mathbf{0}, b, \mathbf{1}, b'$ coincides with the restriction of an affine function h. Moreover, $h(a) + h(a') = 2h((a + a')/2) = 2h((b + b')/2) = 2f((b + b')/2) = f(b) + f(b') \geq f(a) + f(a')$. And we can choose h such that $f(a) \leq h(a)$ and $f(a') \leq h(a')$ hold true. This means that the function f is sub-affine on the octahedron, i.e. f looks alike a *concave* function. Moreover, the rhombus $\mathbf{0}, b, \mathbf{1}, b'$ is an affinity set of f.

In other words, we propagate the function f to the point $\mathbf{1}$ in order to get a concave (discretely) function on the octahedron, such that an affinity area (the convex hull of the affinity set) contains the vector $\mathbf{01}$, the *propagation vector.*

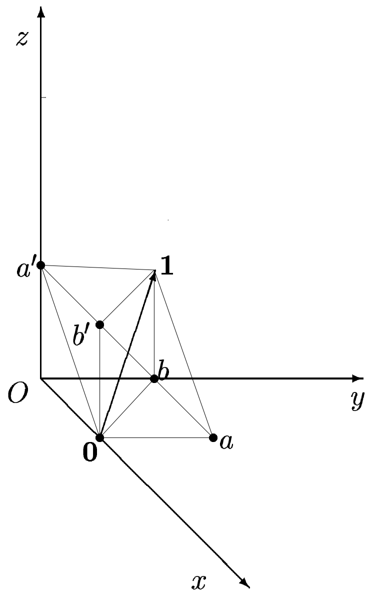
Now, using this rule, which is called the *octahedron recurrence (OR)*, we can propagate a function given at some domain to a large domain. Here is one of possible initial domains (see [13]). Let us consider the set L of points (n, i, j) with integers i, j, n, $n \geq 0$ and $n = i + j \pmod 2$. Suppose a function f is given at a subset of L constituted from points of the form $(\cdot, \cdot, 0)$ and $(\cdot, \cdot, 1)$. Then using the octahedron recurrence with the propagation vector $(0, 0, 2)$ we can propagate the function to points of L of the form $(\cdot, \cdot, 2)$, and so on to the whole L.

Of course, the initial data can be given at more sophisticated subsets, see [13] and [9].

We will display the octahedron recurrence in a slightly different manner[2]. Namely, for us it will be convenient to consider the octahedron recurrence with the propagation vector $(-1, 1, 1)$ and locate the initial data at the qudrants Oxy (*ground*) and Oxz (*front wall*). The modular flats take the form $x = a$, $y = b$, $z = c$ and $x + y + z = d$, where $a, b, c, d \in \mathbb{Z}$. If we cut $\mathbb{R}^3$ by these planes, we get a decomposition of $\mathbb{R}^3$ into primitive tetrahedrons and octahedrons. All octahedrons are parallel, that is one can be obtained by an integer translation of another.

In this set-up, the unit octahedron is of the form depicted at Picture 2.

[2]Sometimes it is convenient to draw pictures for the OR with different propagation vectors. In order to set the octahedron recurrence, we have to choose a unimodular set in the lattice $\mathbb{Z}^3$, say, $\{e_1, e_2, e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3\}$ and the propagation vector $e_3 - e_1 + e_2$, where e_1, e_2, e_3 is a basis in the lattice $\mathbb{Z}^3$. Then the primitive octahedron becomes the convex hull of the points 0, $e_3 - e_1$, e_2, e_3, $e_2 - e_1$, $e_3 - e_1 + e_2$. An integer translation of a plane, spanned by a triple of vectors in the unimodular set, is a *modular flat.* A *non-modular flats* are parallel to planes spanned either by the pair $(e_3 - e_1, e_2)$ or $(e_2 - e_1, e_3)$.



Picture 2.

Thus, a primitive propagation takes the following form: given values at the points **0**, a and b at the ground flour and two values at the points a' and b' at the first flour, due to the OR we get a value at the third point **1** at the first flour.

We claim that functions, which we get as an output of the octahedron recurrence, inherit some concavity properties of input functions. The next two sections are devoted to this issue.

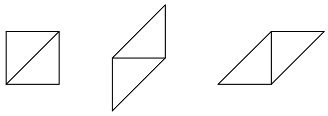
3 Discrete concave functions on 2D-grids

We consider functions on $\mathbb{Z}^2$ defined on finite sets of special form. We call such sets grids and they are specified as follows. A finite subset $T \subset \mathbb{Z}^2$ is a *grid* if i) T has no holes, i.e. $T = co(T) \cap \mathbb{Z}^2$, and ii) any edge of the convex hull $co(T)$ is parallel to one of the vectors $(1,0)$, $(0,1)$ $(1,1)$. (Obviously, a grid has a hexagonal shape, which might degenerated to a pentagon, a trapezoid, a parallelogram or a triangle.)

Let $f : T \to \mathbb{R}$ be a function on a grid T. A primitive triangle in T is either a triple x, $x + (0,1)$ and $x + (1,1)$ of points of T, or a triple x, $x + (1,0)$, $x + (1,1)$. Convex hulls of these primitive triangles constitute a simplicial decomposition of $\mathrm{co}(T)$ (if T is not one-dimensional). We uniquely interpolate the function f by affinity to the triangles on this decomposition of $\mathrm{co}T$, and get a function $\tilde{f} : co(T) \to \mathbb{R}$.

Definition. A function f on a grid T is said to be *discrete concave*, if the interpolation $\tilde{f}$ is a concave function on $\mathrm{co}(T)$.

We can reformulate discrete concavity of a function f without using the interpolation $\tilde{f}$. Namely we have to require validity of three types of "rhombus" inequalities. Consider "primitive" rhombus in T of the form



Then discrete concavity is equivalent to validity of three types of "rhombus" inequalities. The inequalities require that sum at two points of drawn diagonal is greater or equal to the sum at two points of non-drawn diagonal.

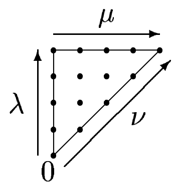
(i) $f(i,j) + f(i+1,j+1) \geq f(i+1,j) + f(i,j+1)$;

(ii) $f(i,j+1) + f(i+1,j+1) \geq f(i+1,j+2) + f(i,j)$;

(iii) $f(i+1,j) + f(i+1,j+1) \geq f(i,j) + f(i+2,j+1)$.

Note, that if only the requirement (i) is valid, then a function is called *supermodular*. If a function is supermodular and the requirement (ii) is valid, then the function is discrete concave on every vertical strip of the unit length, and we call such functions *vertically-strip concave* (VS-concave). Analogously, if (i) and (iii) are valid, a function is called *horizontally strip-concave* (HS-concave).

Mostly, we will be interested in functions on the triangle grid with the vertexes $(0,0)$, $(0,n)$, (n,n); denoted by Δ_n. On the next picture we depicted the grid Δ_4.



Consider a discrete concave function f on the grid Δ_n[3] and consider its restriction to each side of the triangle: the left-hand side, the top of the triangle and the hypotenuse. Specifically, we orient these sides as depicted on the previous picture and consider increments of the function on each unit segment. Then, increments along the left-hand side constitute an n-tuple

$$\lambda(1) = f(0,1) - f(0,0), \lambda(2) = f(0,2) - f(0,1), ..., \lambda(n) = f(0,n) - f(0,n-1).$$

It is easy follows from the rhombus inequalities of the type (i) and (iii) that

$$\lambda(1) \geq \lambda(2) \geq ... \geq \lambda(n).$$

Analogously, we define n-tuple μ ($\mu(i) = f(i,n) - f(i-1,n)$, $i = 1,...,n$) and ν ($\nu(k) = f(k,k) - f(k-1,k-1)$, $k = 1,...,n$), which are also decreasing tuples. We call these n-tuples *increments* of the function f on the corresponding sides of the triangle grid. Obviously, the increments are invariant

[3] Such a discrete concave function was called a *hive* in [11, 12].

under adding a constant to f. Therefore, we have to consider functions modulo adding a constant or to require $f(0,0) = 0$.

Let us briefly say about main roles of discrete concave functions in combinatorics and representation theory. We let to denote $DC_n(\lambda, \mu, \nu)$ the set of discrete concave functions on the grid Δ_n with increments λ, μ, ν. This set is a polytope (probably empty) in the space of all functions on Δ_n. If this polytope is non-empty, when the n-tuples λ, μ, ν are decreasing and there holds $|\lambda| + |\mu| = |\nu|$. For $n > 2$, we need more relations in order to get a non-empty $DC_n(\lambda, \mu, \nu)$. The necessary and sufficient conditions for non-emptyness of $DC_n(\lambda, \mu, \nu)$ (so-called Horn inequalities) are in [11], see also [8, 10, 3]. Moreover, $DC_n(\lambda, \mu, \nu)$ is non-empty if and only if there exist Hermitian matrices A and B, such that A, B, $A + B$ have spectra λ, μ, ν, respectively (a solution to the Horn problem).

We let to denote $DC_n^{\mathbb{Z}}(\lambda, \mu, \nu)$ the set of integer-valued discrete concave functions on the grid Δ_n, of course the tuples λ, μ, ν have to be integer-valued as well. The cardinality of this set coincides with the Littlewood-Richardson coefficient, the multiplicity of the irreducible representation V_ν (of $GL(n)$) in the tensor product irreps $V_\lambda \otimes V_\mu$.

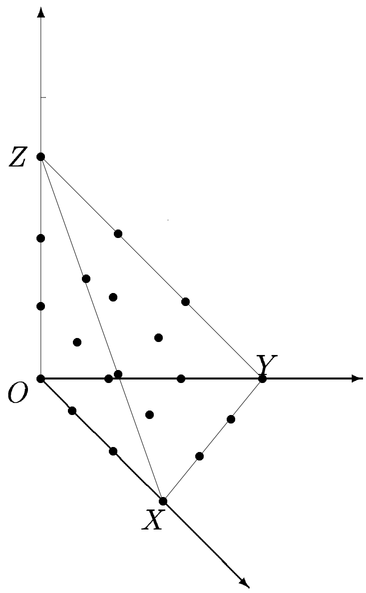
4 Functions on 3D-grids

Recall that we consider the octahedron recurrence with the propagation vector $(-1, 1, 1)$ and locate the initial data at the qudrants OXZ (*ground*) and OXY (*front wall*). The modular flats take the form $x = a$, $y = b$, $z = c$ and $x+y+z = d$, where $a, b, c, d \in \mathbb{Z}$. All primitive octahedrons are parallel, and each octahedron has three diagonals parallel to vectors $(1, 1, -1)$, $(1, -1, 1)$ and $(-1, 1, 1)$, respectively, and corresponding three pairs of antipodal vertexes. Any two of this diagonal vectors span a non-modular flat.

The diagonal being parallel to the propagation vector $(-1, 1, 1)$, we call the *mail* diagonal.

Definition. A function $F : \mathbb{Z}^3 \to \mathbb{R} \cup \{-\infty\}$ is said to be *polarized*, if, for any primitive octahedron, sum of values of F at the vertexes of the main diagonal is equal to the maximum of the sum of values of F at the antipodal vertexes of two others diagonals.

We denote $\Delta_n(OXYZ)$ the three-dimensional grid, constituted from the non-negative integer points (x, y, z), such that $x + y + z \leq n$ (see Picture 3 with $\Delta_3(XYZ)$). It is easy to see that, for any initial data given at the *ground* $\Delta_n(OXY)$ and the *front wall* $\Delta_n(OXZ)$, there exists a unique polarized function with domain $\Delta_n(OXYZ)$ and these initial data. This is due to the OR. However, we can set initial data at the *shadow wall* $\Delta_n(OYZ)$ and the *slope wall* $\Delta_n(XYZ)$ and get a polarized function. In that case, we have to apply the OR with the reverse propagation vector $(1, -1, -1)$.



Picture 3.

The fundamental property of the octahedron recurrence is that if the initial data (at the ground and the front wall) are discrete concave function, then the corresponding polarized function on the grid $\Delta_n(OXYZ)$ is a kind of three-dimensional discrete concave function. Without going in details of discrete concave functions in $\mathbb{Z}^n$, we give relevant notions for purposes this paper.

Discrete concavity on **2D**-grids is equivalent to fulfilling of three kinds of rhombus inequalities. In dimension 3, we have four kinds of modular flats. In each such a 2-dimensional flat we have rhombuses, which corresponds to triangular decomposition of the flat by cutting it by three others kinds of modular flats. We require validity of the rhombus inequality for each such a rhombus in each flat: the sum of values at the "short" diagonal is greater or equal to the sum at the "long" diagonal.

Definition. A function $F : \mathbb{Z}^3 \to \mathbb{R} \cup \{-\infty\}$ is a *polarized discrete concave* function if F is polarized and all kinds of rhombus inequalities in each modular flat are fulfilled.

Let us denote by PDC_n the set of polarized discrete concave functions on the three-dimensional grid $\Delta_n(OXYZ)$.

Theorem 1. *Let F be a polarized function on the three-dimensional grid $\Delta_n(OXYZ)$. Suppose the restriction of F to the ground face $\Delta_n(OXY)$ and to the front wall face $\Delta_n(OXZ)$ are 2-dimensional discrete concave functions. Then $F \in PDC_n$.*

For a proof see [6]. Note, that this theorem is equivalent to the following corollary (a sketch of proof of which is in [9]).

Corollary 1. *If the restrictions of a polarized function to the ground and the front wall faces are discrete concave, then the restriction to the shadow wall and the slope wall are also discrete concave.*

Proof. In fact, any rhombus located on the slope or shadow wall is also a rhombus for three-dimensional grid, and, therefore, the corresponding rhombus inequality is valid. □

Corollary 2 *Let the restriction of a polarized function to the ground be discrete concave and the restriction to the front wall be HS-concave. Then the restrictions to two other faces are HS-concave.*

Proof. In fact, we can add to F an appropriate function $\varphi(z)$ of the vertical variable z, in order to get a discrete concave function on the front wall. $F + \varphi(z)$ is not changed on the ground, therefore by Corollary 1, $F+\varphi(z)$ is discrete concave at the other two wall, therefore F is HS-concave on these walls. □

Corollary 3. *Suppose the restriction of a polarized function to the ground is HS-concave and the restriction to the front wall is VS-concave (here we consider horizontal being parallel to the segment XY). Then F is VS-concave on the shadow wall.*

Proof. As above, having add to F an appropriate separable function on variables x and y, we get a polarized function $G = F + \varphi(x) + \psi(y)$, which will be discrete concave on the ground and the front wall. By Corollary 1, G is discrete concave on the shadow wall. Therefore, F is VS-concave on this wall. □

Corollary 4. *Suppose a polarized function F is discrete concave on the ground and VS-concave on the front wall. Then F is discrete concave on the shadow wall.*

Proof. In fact, having add an appropriate function on x to F, we get a discrete concave function on the front wall. On the ground this function will be also discrete concave. But this function remains the same on the shadow wall, and by Corollary 1 the function on this wall is discrete concave. □

Now let us consider the polarized functions (or the octahedron recurrence) on the prism $\Delta_n(OXY) \times \{0, 1, ..., m\}$.

At points outside the prism, we set functions equal $-\infty$. Therefore, on the non-modular face $\Delta_n(XY) \times \{0, 1, ..., m\}$, a polarized function F turns out to be a separable function (on variables $x + y$ and z).

Corollary 5 *Let F be a polarized function on the prism $\Delta_n(OXY) \times \{0, 1, ..., m\}$. Suppose the restriction of F to the ground face $\Delta_n(OXY) \times \{0\}$ and the restriction to the front wall $\Delta_n(OX) \times \{0, 1, ..., m\}$ are discrete concave functions. Then F is polarized discrete concave function on the prism* *(and, in particular, F is discrete concave on the shadow wall $\Delta_n(OY) \times \{0,1,...,m\}$ and on the ceiling $\Delta_n(OXY) \times \{m\}$).*

Proof. It is easy to see that it suffices to prove the corollary in the case $m = 2$.

In the beginning we consider the case $m = 1$. Let us extend the ground to the size of $n+1$, that is we add to $\Delta_n(OXY)$ new points $(n+1,0,0),...,(0,n+1,0)$. Let us extend F to these points such that we get a discrete concave function on the extended ground $\Delta_{n+1}(OXY) \times \{0\}$ and a discrete concave function on the "extended" front wall. We can always to do that by setting small values ($<< 0$) to these new points. Let us denote $\tilde{F}$ this extension. By Theorem 1, the function $\tilde{F}$ is a polarized discrete concave function. We claim, that the restriction of this function to the prism is a polarized discrete concave function. In fact, it suffices to check that $\tilde{F}$ coincides with F on the non-modular face $\Delta_n(XY) \times \{0,1\}$. But this holds since we assigned small values to the new points. Thus F and $\tilde{F}$ coincide on the prism. Since $\tilde{F}$ is discrete concave function, F is discrete concave too.

Let us move to the case $m = 2$. We have to check all rhombus inequalities for all rhombuses in the prism $\Delta_n(OXY) \times \{0,2\}$. Let us first consider the rhombuses of the vertical size 2. It is easy to see that these rhombuses belong to the tetrahedron of size $n+1$. Then the corresponding rhombus inequality is valid, since they are valid for $\tilde{F}$. Other rhombuses are located either in the prism $\Delta_n(OXY) \times \{0,1\}$, or in the prism $\Delta_n(OXY) \times \{1,2\}$. For the first prism, the corresponding inequality follows due to the above case with $m = 1$. Moreover, we get that F is discrete concave on the triangle $\Delta_n(OXY) \times \{1\}$. Now, again applying the case $m = 1$ to the prism $\Delta_n(OXY) \times \{1,2\}$, we get validity of rhombus inequalities in this prism. □

Using similar reasonings one can get the following

Corollary 6 *Suppose a polarized function F on the prism $\Delta_n(OXY) \times \{0,1,...,m\}$ has discrete concave restrictions to the ceiling $\Delta_n(OXY) \times \{m\}$ and the shadow wall $\Delta_n(OY) \times \{0,1,...,m\}$. Then F is a polarized discrete concave function and its restrictions to the front wall $\Delta_n(OX) \times \{0,1,...,m\}$ and the ground $\Delta_n(OXY) \times \{0\}$ are discrete concave functions.*

5 Functional form of RSK

Consider the rectangle $[0,n] \times [0,m]$ on the plane with natural n and m, constituted from unit squares with the centers at the points $(i-1/2, j-1/2)$, $i = 1,...,n$, $j = 1,...,m$, we call such squares boxes. An array is a filling of each box (i,j) with a non-negative "mass" $a(i,j)$.

To each array a we associate a function $f = f_a$ on the rectangular grid

$\{0,1,...,n\} \times \{0,1,...,m\}$ by setting to the point (i,j) the value

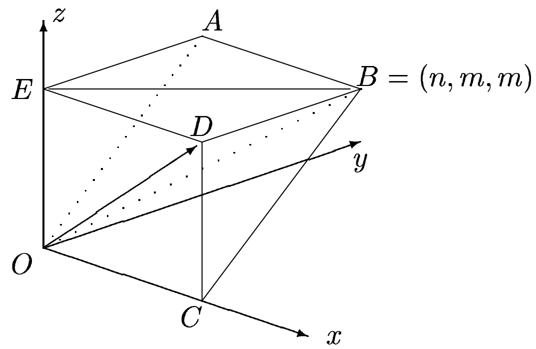
$$f_a(i,j) = \sum_{i' \leq i, j' \leq j} a(i',j').$$

In other words, this value is equal to the mass of all boxes to the south-west to the point (i,j). This is a reason to denote by $\iint a$ the function f_a. On the bottom and the left boundary of the rectangle the function equals 0. For other (i,j), we obviously have

$$f(i,j) - f(i-1,j) - f(i,j-1) + f(i-1,j-1) = a(i,j).$$

From this $a(i,j)$ might be understand as the mixed derivative of f $(a = \partial\partial f)$, or as a break of f along the common edge $[(i-1,j-1),(i,j)]$ of two affinity areas. Since $a(i,j) \geq 0$, the function $\iint a$ is supermodular.

Here it will be convenient to consider the octahedron recurrence with the propagation vector $(1,0,1)$. On Picture 4 with $n = m$ we depicted modular and non-modular flats: the modular flats are parallel to the faces of the tetrahedron $OEAB$, and the non-modular flats parallel to the face $OEDC$ and the plane passing through ODB (the propagation vector is parallel to the vector OD).



Picture 3.

Let us locate the null function at the face OEA; at the slope rectangle $OABC$ we locate the function $f_a = \iint a$. Specifically, we assign the value $\iint a(i,j))$ to the point (i,j,j). Now, we propagate these data by the octahedron recurrence to the prism. From Corollary 5 (Section 4), we get a VS-concave function at the top face rectangle $EABD$ and HS-concave function at the right face triangle CDB (the vertical is the y-axe in the first case, and the z-axe in the second case).

Thus, we have

Theorem 2. *Let a be an array, and let F be a function on the prism obtained by the octahedron recurrence from the following initial data: the zero values at the faces $OEDC$ and OEA, and $\iint a$ at $OABC$. Then the* *restriction of F to $EABD$ is a VS-concave function and the restriction to CDB is an HS-concave function.*

Let us note, that the latter two functions coincide at the edge DB.

Remark. We can also propagate data in the reverse direction. Specifically, assume we are given a function f on the top face $EABD$ and a function g on the triangle CDB. Suppose there hold

a) f is VS-concave and equals 0 at the edges EA and ED;

b) g is HS-concave and equals 0 at the edge CD;

c) the functions f and g coincide at the edge DB.

Then having applied the OR (with the propagation vector $(-1,0,-1)$) to these data, we get a pair of functions on the triangle OEA and the slope rectangle $OABC$. Due to Corollary 6, we get a discrete concave function on OEA and a supermodular function on $OABC$. Moreover, we get the null function on the triangle OEA. This is because this function equals 0 at the edge EA (due to the item a)) and at the edge OE (this follows from b) and separability of the OR on the non-modular face). But these boundary values force nullity of a discrete concave function.

Now, we get an array a as the mixed derivatives of the resulting supermodular function on $OABC$. Thus, this octahedron recurrence provides us with a functional form of the RSK (indeed modified RSK [4]).

Namely, due to a bijection between VS-concave functions (HS-concave) functions and semi-standard Young tableaux ([4]), the above Theorem and Remark establish a bijection between the set of arrays and and the set of pairs of SSYT of equal shape (the increments of the functions along the edge DB is exactly the shape of the tableaux).

Let us briefly explain the mapping from VS-concave functions to SSYT. Namely, let f be a VS-concave function. Let us consider the array $\partial\partial f$. To get the corresponding semi-standard Young tableaux we have to read this array from left to right and from bottom to top. Reading a row gives us filling of the corresponding row in the Young tableau, the mass $a(i,j)$ exhibits the multiplicity of repetitions of the letters i in the j-th row of the Young tableaux (we consider the French style of drawing Young diagrams and tableaux). Consider an example. Let f be given by $\begin{matrix} 5 & 10 & 12 & 23 \\ 5 & 10 & 12 & 20 \\ 5 & 6 & 8 & 12 \end{matrix}$.

Then $\partial\partial f = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 4 & 0 & 4 \\ 5 & 1 & 2 & 4 \end{pmatrix}$, and the corresponding semi-standard Young tableau is

$$\begin{matrix} 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 & 4 \end{matrix}$$

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