# A Structure Theory of the Sandpile Monoid for Digraphs (EXTENDED ABSTRACT)

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**Abstract**

The Abelian Sandpile Model is a diffusion process on graphs, studied, under various names, in statistical physics, theoretical computer science, and algebraic graph theory. The model takes a rooted directed multigraph $\mathcal{X}^*$, the *ambient space*, in which the root is accessible from every vertex, and associates with it a commutative monoid $\mathcal{M}$, a commutative semigroup $\mathcal{S}$, and an abelian group $\mathcal{G}$ as follows. For vertices $i, j$, let $a_{ij}$ denote the number of $i \to j$ edges and let $\deg(i)$ denote the out-degree of $i$ in $\mathcal{X}^*$. Let $V$ be the set of ordinary (non-root) vertices. With each $i \in V$ associate a symbol $x_i$ and consider the relations $\deg(i)x_i = \sum_{j \in V} a_{ij}x_j$. Let $\mathcal{M}$, $\mathcal{S}$, and $\mathcal{G}$ be the commutative monoid, semigroup and group, respectively, generated by $\{x_i : i \in V\}$ subject to these defining relations. $\mathcal{M}$ is the *sandpile monoid*, $\mathcal{S}$ is the *sandpile semigroup*, and $\mathcal{G}$ is the *sandpile group* associated with $\mathcal{X}^*$. We observe that $\mathcal{G}$ is the unique minimal ideal of $\mathcal{M}$.

We establish connections between the algebraic structure of $\mathcal{M}$, $\mathcal{S}$, $\mathcal{G}$, and the combinatorial structure of the underlying ambient space $\mathcal{X}^*$. $\mathcal{M}$ is a distributive lattice of semigroups each of which has a unique idempotent. The distributive lattice in question is the lattice $\mathcal{L}$ of idempotents of $\mathcal{M}$; $\mathcal{L}$ turns out to be isomorphic to the dual of the lattice of ideals of the poset of normal strong components of $\mathcal{X}^*$ (strong components which contain a cycle). The $\mathcal{M} \to \mathcal{L}$ epimorphism defines the *finest* semilattice congruence of $\mathcal{M}$; therefore $\mathcal{L}$ is the *universal semilattice* of $\mathcal{M}$.

We characterize the directed graphs $\mathcal{X}^*$ for which $\mathcal{S}$ has a unique idempotent; this includes the important case when the digraph induced on the ordinary vertices is strongly connected. If the idempotent in $\mathcal{S}$ is unique then the Rees quotient $\mathcal{S}/\mathcal{G}$ (obtained by contracting $\mathcal{G}$ to a zero element) is nilpotent. Let, in this case, $k$ denote the nilpotence class of $\mathcal{S}/\mathcal{G}$. Our main result establishes the existence of functions $\psi_1$ and $\psi_2$ such that $|\mathcal{S}/\mathcal{G}| \le \psi_1(k)$ and $\mathcal{G}$ contains a cyclic subgroup of index $\le \psi_2(k)$. This result is a corollary to our asymptotic characterization of the ambient spaces with bounded $k$: every sufficiently large directed multigraph with this property can be described as a "circular tollway system of bounded effective volume."

## 1 The Abelian Sandpile Model

The Abelian Sandpile Model is a diffusion process on graphs, studied, under various names, in statistical physics, theoretical computer science, and algebraic graph theory[1]. The model takes a finite directed multigraph ("digraph") $\mathcal{X}^*$ with a special vertex called the *sink* as its *ambient space*, and associates with it a finite commutative monoid $\mathcal{M}$, a finite commutative semigroup $\mathcal{S}$, and a finite abelian group $\mathcal{G}$, called the *sandpile monoid*, the *sandpile semigroup* and the *sandpile group* of $\mathcal{X}^*$, respectively. We assume that the sink is accessible from every vertex and has out-degree zero. Vertices other than the sink will be called *ordinary*. A *state* of the game is an assignment of an integer $h_i \ge 0$ to each ordinary vertex $i$. The integer $h_i$ may

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[1]The Abelian Sandpile Model is identical with the "dollar-game" in Biggs' terminology [3], and is a variant of the "chip-firing game" studied in computer science [5].

be thought of as the number of sandgrains (the *height* of the sandpile) at *site* $i$. A state is *stable* if for all ordinary vertices $i$, $0 \le h_i < \deg(i)$, where deg denotes the out-degree in $\mathcal{X}^*$. If $h_i \ge \deg(i)$ for an ordinary vertex $i$, the "pile" at $i$ may be "toppled," sending one grain through each edge leaving $i$. So $h_i$ is reduced by $\deg(i)$, and for each ordinary vertex $j$, the height $h_j$ increases by $a_{ij}$, the number of edges from $i$ to $j$. The sink "collects" the grains "falling off" the ordinary vertices and never topples. Starting with any state and toppling unstable ordinary vertices in succession, we arrive at a stable state in a finite number of steps, since the sink is accessible from every vertex.

By a Jordan-Hölder argument (the "Diamond Lemma," cf. [14]), the order in which the topplings occur does not matter [5, 7]; given an initial state $\mathbf{h}$, every stabilizing sequence ("avalanche") leads to the same stable state $\sigma(\mathbf{h})$; hence the term "abelian."

## 2 The Sandpile Monoid

Our standard reference to semigroup theory is Grillet [9].

The *sandpile monoid* is defined as the set of stable states under the operation of pointwise addition and stabilization. We denote this operation by $\oplus$. So, for stable states $\mathbf{h}_1$ and $\mathbf{h}_2$ we set $\mathbf{h}_1 \oplus \mathbf{h}_2 := \sigma(\mathbf{h}_1 + \mathbf{h}_2)$. The all-zero state 0 is the identity in $\mathcal{M}$. The subsemigroup of $\mathcal{M}$ generated by the non-zero states is the *sandpile semigroup* $\mathcal{S}$. Clearly, $\mathcal{M} = \mathcal{S} \cup \{0\}$.

## 3 The Sandpile Group

**Fact 3.1** *Every finite monoid has a unique minimal ideal.*

**Fact 3.2** *The minimal ideal of a finite commutative monoid is a group.*

**Definition 3.3** The unique minimal ideal of the sandpile monoid is called the *sandpile group*.

Note that this definition is more concise but equivalent to the definitions occurring in the literature (Dhar [7], Creutz [6]).

The common approach to defining the sandpile group is the following: We say that the stable state $\mathbf{h}_1$ is *accessible* from the stable state $\mathbf{h}_2$ if $(\exists \mathbf{h} \in \mathcal{M})(\mathbf{h}_1 = \mathbf{h}_2 \oplus \mathbf{h})$. We say that a state $\mathbf{h}_1$ is accessible from a state $\mathbf{h}_2$ if $\sigma(\mathbf{h}_1)$ is accessible from $\sigma(\mathbf{h}_2)$. A stable state is called *recurrent* (or "critical") if it is accessible from every state. The set of recurrent states is then defined to be the sandpile group. (It is identical with the unique minimal ideal of $\mathcal{M}$.)

## 4 Generators and relations for the Sandpile Monoid

Let us fix some notation. The digraph $\mathcal{X}^* = (V^*, E^*)$ denotes our ambient space; a vertex is designated as the sink; recall that $a_{ij}$ is the number of edges from vertex $i$ to vertex $j$. Finally, $\mathcal{X} = (V, E)$ denotes the subgraph of $\mathcal{X}^*$ induced on the set $V$ of ordinary (non-sink) vertices.

**Proposition 4.1**

*(i) The sandpile monoid is the commutative monoid generated by the symbols $\{x_i : i \in V\}$ subject to the set of defining relations $\mathcal{R} = \{\deg(i)x_i = \sum_{j \in V} a_{ij}x_j : i \in V\}$.*

*(ii) The sandpile semigroup is the commutative semigroup generated by the symbols $\{x_i : i \in V\}$ subject to the set of defining relations $\mathcal{R} = \{\deg(i)x_i = \sum_{j \in V} a_{ij}x_j : i \in V\}$.*

# 5 Generators and relations for the Sandpile Group

**Definition 5.1** Let $\mathcal{M}$ be a monoid, $\mathcal{G}$ a group and $\phi : \mathcal{M} \to \mathcal{G}$ a homomorphism. We say that $(\phi, \mathcal{G})$ is the *universal group* of $\mathcal{M}$ if every homomorphism from $\mathcal{M}$ to a group factors through $\phi$.

**Observation 5.2** *If $\langle W|R\rangle$ is a presentation of a monoid $\mathcal{M}$, then $\langle W|R\rangle$ is also a presentation of the universal group $\mathcal{G}$ of $\mathcal{M}$ as a group.*

**Fact 5.3** *Let $\mathcal{M}$ be a finite commutative monoid and let $\mathcal{G}$ be the minimal ideal of $\mathcal{M}$. Let $e \in \mathcal{G}$ be the identity in $\mathcal{G}$ and let $\phi : \mathcal{M} \to \mathcal{G}$ be defined by $\phi(x) := e + x$. Then $(\phi, \mathcal{G})$ is the universal group of $\mathcal{M}$.*

**Corollary 5.4** *The sandpile group is the universal group of the sandpile monoid (under the homomorphism described in Fact 5.3).*

**Corollary 5.5 (Dhar[7])** *The sandpile group is isomorphic to the quotient $\mathbb{Z}^V/\Lambda$, where $\Lambda$ is the lattice spanned by the rows of the reduced Laplacian (see Definition 5.7).*

**Definition 5.6** The *Laplacian* $L = (L_{ij})_{i,j\in V^*}$ of $\mathcal{X}^*$ is the $|V^*| \times |V^*|$ matrix defined by

$$L_{ij} := \begin{cases} \deg(i) - a_{ii} & \text{if } i = j, \\ -a_{ij} & \text{otherwise.} \end{cases} \tag{1}$$

**Definition 5.7** The *reduced Laplacian* $\Delta = (\Delta_{ij})_{i,j\in V}$ of $\mathcal{X}^*$ is defined as the matrix obtained from the Laplacian $L$ by deleting the row and the column corresponding to the sink.

The digraph version of Kirchhoff's [12] classical Matrix-Tree Theorem (Tutte [17], cf. [13]) now implies:

**Corollary 5.8 (Dhar[7])** *The order of the sandpile group is the number of directed spanning trees of the ambient space $\mathcal{X}^*$ directed towards the sink.*

# 6 The universal lattice of the Sandpile Monoid

**Definition 6.1** Let $\mathcal{S}$ be a semigroup, $\mathcal{L}$ a semilattice, and $\phi : \mathcal{S} \to \mathcal{L}$ a homomorphism. We say that $(\phi, \mathcal{L})$ is the *universal semilattice* of $\mathcal{S}$ if every homomorphism from $\mathcal{S}$ to a semilattice factors through $\phi$.

**Fact 6.2** *Every semigroup has a universal semilattice.*

**Fact 6.3** *For a finite monoid, the universal semilattice is a lattice.*

**Fact 6.4** *Every finite lattice is the universal lattice of a finite commutative monoid (namely, of itself).*

**Fact 6.5** *For a finite commutative monoid $\mathcal{M}$, the universal lattice is isomorphic to the lattice of idempotents of $\mathcal{M}$ and $\mathcal{M}$ is a lattice of semigroups with unique idempotent.*

**Theorem 6.6** *For a finite lattice $\mathcal{L}$, the following are equivalent:*

*(i) $\mathcal{L}$ is the universal lattice of a sandpile monoid.*

*(ii) $\mathcal{L}$ is distributive.*

The proof of Theorem 6.6 goes through a description of the semilattice of idempotents of the sandpile monoid in terms of the strong components of $\mathcal{X}$:

**Definition 6.7**

(i) A vertex is *normal* if it belongs to a cycle; and *abnormal* otherwise.

(ii) A strong component of a digraph is *normal* if it contains a cycle.

So, an abnormal strong component consists of a single abnormal vertex. All vertices of a normal strong component are normal.

**Theorem 6.8** *The following lattices are isomorphic:*

*(i) The lattice of idempotents of the sandpile monoid corresponding to the ambient space $\mathcal{X}^*$.*

*(ii) The dual lattice of ideals of the accessibility partial order on the set of normal strong components of $\mathcal{X}$.*

**Corollary 6.9** *The sandpile semigroup has a unique idempotent if and only if*

*(i) $\mathcal{X}$ is a directed acyclic graph (DAG) with at least one vertex of* $\deg \geq 2$ (*degree, as always, relative to $\mathcal{X}^*$), or*

*(ii) $\mathcal{X}$ has a unique normal strong component.*

# 7 Bounded nilpotence class

**Fact 7.1** *If $\mathcal{S}$ is a finite semigroup with a unique idempotent and $\mathcal{G}$ is the minimal ideal of $\mathcal{S}$ then the Rees quotient (obtained by contracting $\mathcal{G}$ to a zero element) is nilpotent.*

In this section we consider the case where the sandpile semigroup $\mathcal{S}$ has a unique idempotent. This is the case, in particular, when $\mathcal{X}$ is strongly connected.

Let $k$ denote the nilpotence class of $\mathcal{S}/\mathcal{G}$. We observe that $k-1$ is the maximum weight of any (not necessarily stable) transient (non-recurrent) state $\mathbf{h} \in \mathbb{N}^V$.

We need to treat DAGs separately.

**Proposition 7.2** $\mathcal{M} = \mathcal{G}$ *if and only if $\mathcal{X}$ is a DAG.*

Our main result (Theorem 7.10) will asymptotically characterize the ambient spaces corresponding to bounded nilpotence class. The main corollary to the result asserts, somewhat surprisingly, that the boundedness of the nilpotence class of $\mathcal{S}/\mathcal{G}$ implies the boundedness of $\mathcal{S}/\mathcal{G}$ itself, and has strong structural implications on the sandpile group $\mathcal{G}$.

**Corollary 7.3** *There exist functions $\psi_1$ and $\psi_2$ such that if the sandpile quotient $\mathcal{S}/\mathcal{G}$ has nilpotence class $k$ then its order is $|\mathcal{S}/\mathcal{G}| \leq \psi_1(k)$; and if $\mathcal{X}$ is not a DAG then the sandpile group $\mathcal{G}$ contains a cyclic subgroup of index $\leq \psi_2(k)$.*

**Definition 7.4**

(i) We define the *strong degree* of the vertex $v$ (denoted by $\deg_s(v)$) to be the number of edges from $v$ to the vertices in the strong component of $v$. So, an ordinary vertex $v$ is abnormal if and only if $\deg_s(v) = 0$.

(ii) Let $\mathcal{X}^*$ be an ambient space and let $A$ be the set of vertices that belong to all cycles in $\mathcal{X}$. We define the *effective volume* of $\mathcal{X}^*$ to be

$$\operatorname{vol}(\mathcal{X}^*) := \prod_{v \in A} \deg_s(v) \prod_{v \in V \setminus A} \deg(v). \tag{2}$$

**Theorem 7.5** *For a class of ambient spaces $\mathcal{C}$, the following are equivalent:*

*(i) The nilpotence class of the Rees quotients $\mathcal{S}(\mathcal{X}^*)/\mathcal{G}(\mathcal{X}^*)$ is bounded ($\mathcal{X}^* \in \mathcal{C}$).*

*(ii) The number of transient states, $|\mathcal{M}(\mathcal{X}^*) \setminus \mathcal{G}(\mathcal{X}^*)|$, is bounded ($\mathcal{X}^* \in \mathcal{C}$).*

*(iii) The effective volume* $\mathrm{vol}(\mathcal{X}^*)$ *is bounded ($\mathcal{X}^* \in \mathcal{C}$).*

Now we describe the asymptotic structure of $\mathcal{X}^*$ for bounded $k$.

**Definition 7.6** Let $\mathcal{Y}$ be a directed acyclic graph (DAG) and let $v$ be a vertex.

(i) The vertex $v$ is an *entrance* if every vertex of $\mathcal{Y}$ is accessible from $v$.

(ii) The vertex $v$ is an *exit* if $v$ is accessible from every vertex of $\mathcal{Y}$.

(iii) $\mathcal{Y}$ is a *rest area* if it has an entrance and an exit. Note that the entrance and the exit are unique.

(iv) The *interior* vertices of a rest area are the vertices other than the entrance and the exit. The set of interior vertices of the rest area $\mathcal{Y}$ is denoted by $\mathrm{int}(\mathcal{Y})$.

**Definition 7.7** We say that an ordinary vertex $v \in V$ is *thin* if $\deg_s(v) = 1$.

**Definition 7.8**

(i) A *circular highway* is a strong component of $\mathcal{X}$ which is a thin directed cycle (every vertex of the cycle is thin).

(ii) We now construct a *circular tollway.* We start with a circular highway on which we designate edges $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$ in this cyclic order ($n \geq 0$). For $i = 1, \ldots, n$, we delete the edge $(u_i, v_i)$ and we glue a rest area $R_i$ between $u_i$ and $v_i$ so that $\mathrm{entrance}(R_i) = u_i$ and $\mathrm{exit}(R_i) = v_i$. (The $R_i$ are disjoint.)

(iii) We say that an ambient space $\mathcal{X}^*$ is a *circular tollway system* if $\mathcal{X}$ has a unique normal strong component and this strong component is a circular tollway.

**Definition 7.9** Let $\mathcal{X}^*$ be an ambient space.

(i) We call an ordinary vertex $v$ *relevant* if $\deg(v) \geq 1$.

(ii) An edge $(v, w)$ is *relevant* if its tail $v$ is a relevant vertex. (In this definition, $v \in V$ and $w \in V^*$.)

**Theorem 7.10** *For a class of ambient spaces $\mathcal{C}$, the following are equivalent:*

*(a) Any of the equivalent conditions (i), (ii) and (iii) of Theorem 7.5.*

*(b) $(\exists n_0 \geq 0)$(if $\mathcal{X}^* \in \mathcal{C}$ has more than $n_0$ relevant edges then $\mathcal{X}^*$ is a circular tollway system of bounded effective volume).*

We defer the proofs to an expanded version of this paper.

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