COMBINATORIAL INVARIANCE OF KAZHDAN-LUSZTIG POLYNOMIALS FOR SHORT INTERVALS IN THE SYMMETRIC GROUP

FEDERICO INCITTI

ABSTRACT. The well-known *combinatorial invariance conjecture* states that, given a Coxeter group W, ordered by Bruhat order, and given two elements $x, y \in W$, with $x < y$, the Kazhdan-Lusztig polynomial, or equivalently the R-polynomial, associated with (x, y) supposedly depends only on the poset structure of the interval $[x, y]$.

In this paper we solve the conjecture for the first open cases, showing that it is true for intervals of length 5 and 6 in the symmetric group. The main tool is a pictorial way for describing the Bruhat order in the symmetric group, namely the *diagram* of a pair of permutations. It is shown how the diagram of (x, y) allows to get information about the poset structure of $[x, y]$, and about the R-polynomial associated with (x, y). As a parallel result, we obtain expressions of the R-polynomials for some general classes of pairs of permutations.

1. INTRODUCTION

In their fundamental paper [10] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials with integer coefficients, indexed by pairs of elements of W. These polynomials, known as *Kazhdan-Lusztig polynomials* of W, are related to the algebraic geometry and topology of Schubert varieties. They also play a crucial role in representation theory. In order to prove the existence of these polynomials Kazhdan and Lusztig used another family of polynomials which arises from the multiplicative structure of the Hecke algebra associated with W. These polynomials are known as *R-polynomials* of W.

A famous conjecture concerning Kazhdan-Lusztig polynomials, the so-called *combinatorial invariance conjecture*, states that, given two Coxeter groups W_1 and W_2, partially ordered by Bruhat order, and given elements $x, y \in W_1$, with $x < y$, and $u, v \in W_2$, with $u < v$, if the two intervals $[x, y]$ and $[u, v]$ are isomorphic as posets, then $P_{x,y}(q) = P_{u,v}(q)$. This conjecture is known to be true if $[x, y]$ is a lattice and holds for intervals up to length 4. Recently it has been proved that it is true for $x = u = e$ (see [3]).

In this paper we solve the conjecture for the first open cases, showing that it is true for intervals of length 5 and 6 in the symmetric group. The main tool is a pictorial way for describing the Bruhat order in the symmetric group, namely the *diagram* of a pair of permutations.

In §2 we collect some basic notions and in §3 we state the main result. Then we introduce the diagram of a pair of permutations (§4). We develop this tool, already introduced in [9], showing how the diagram of (x, y) allows to get information about the poset structure of $[x, y]$ (§5), and about the R-polynomial associated with (x, y) (§6). As a parallel result, we obtain expressions of the R-polynomials for some general classes of pairs of permutations. After a classification of the intervals of the symmetric group up to length 5 (§7), we give the proof of the main result in §8.

2. PRELIMINARIES

We let $\mathbf{N} = \{1, 2, 3, \dots\}$. For $n \in \mathbf{N}$ we let $[n] = \{1, 2, \dots, n\}$ and for $n, m \in \mathbf{N}$, with $n \le m$, we let $[n, m] = \{n, n + 1, \dots, m\}$. We refer to [11] for poset theory. Given a poset P, we denote by $\lhd$ the covering relation. The *Hasse diagram* of P is the directed graph having P as vertex set and such that there is an edge from x to y if and only if $x \lhd y$. Given $x, y \in P$, with $x < y$, we set $[x, y] = \{z \in P : x \le z \le y\}$, and call it an *interval* of P. We refer to [8] for basic notions about Coxeter groups. Given a Coxeter group W, with set of generator S, the set of *reflections* is

$$T = \{wsw^{-1} : w \in W, \ s \in S\}.$$

Given $x \in W$, the *length* of x, denoted by $\ell(x)$, is the minimal k such that x can be written as a product of k generators. The *Bruhat graph* of W, denoted by $BG(W)$ (or simply BG) is the directed graph having W as vertex set and such that there is an edge $x \to y$ if and only if $y = xt$, with $t \in T$, and $\ell(x) < \ell(y)$. The edge is often supposed labelled by the reflection t:

$$x \xrightarrow{t} y.$$

Finally, the *Bruhat order* of W is the partial order which is the transitive closure of BG: given $x, y \in W$, $x \leq y$ in the Bruhat order if and only if there is a chain $x = x_0 \to x_1 \to \cdots \to x_k = y$. It is known that W, partially ordered by the Bruhat order, is a graded poset with rank function given by the length. Given $x, y \in W$, with $x < y$, we set $\ell(x, y) = \ell(y) - \ell(x)$.

We now define R-polynomials and Kazhdan-Lusztig polynomials.

Theorem 2.1. *There exists a unique family of polynomials $\{R_{x,y}(q)\}_{x,y \in W} \subseteq \mathbb{Z}[q]$ satisfying the following conditions:*

(1) $R_{x,y}(q) = 0$, *if* $x \not\leq y$;
(2) $R_{x,y}(q) = 1$, *if* $x = y$;
(3) *if* $x < y$ *and* $s \in S$ *is such that* $ys \lhd y$ *then*

$$R_{x,y}(q) = \begin{cases} R_{xs,ys}(q), & \text{if } xs \lhd x, \\ (q-1)R_{x,ys}(q) + qR_{xs,ys}(q), & \text{if } xs \rhd x. \end{cases}$$

The existence of such a family is a consequence of the invertibility of certain basis elements of the Hecke algebra $\mathcal{H}$ of W and is proved in [8, §§7.4, 7.5]. The polynomials whose existence and uniqueness are guaranteed by Theorem 2.1 are called the *R-polynomials* of W.

Theorem 2.2. *There exists a unique family of polynomials $\{P_{x,y}(q)\}_{x,y \in W} \subseteq \mathbb{Z}[q]$ satisfying the following conditions:*

(1) $P_{x,y}(q) = 0$, *if* $x \not\leq y$;
(2) $P_{x,y}(q) = 1$, *if* $x = y$;
(3) *if* $x < y$ *then* $\deg(P_{x,y}(q)) < \ell(x,y)/2$ *and*

$$q^{\ell(x,y)} P_{x,y}\left(q^{-1}\right) - P_{x,y}(q) = \sum_{x < z \leq y} R_{x,z}(q)\, P_{z,y}(q).$$

A proof of Theorem 2.2 appears in [8, §§7.9, 7.10, 7.11]. The polynomials whose existence and uniqueness are guaranteed by Theorem 2.2 are called the *Kazhdan-Lusztig polynomials* of W. Theorem 2.2 ensures that knowing the R-polynomials is equivalent to knowing the Kazhdan-Lusztig polynomials. In fact part (3) can be recursively used to compute one family from the other, by induction on $\ell(x, y)$.

The R-polynomials satisfy the following relation (see, e.g., [1, Exercise 5.11]).

Proposition 2.3. *Let $x, y \in W$, with $x < y$. Then*

$$\sum_{x \leq z \leq y} (-1)^{\ell(x,z)} R_{x,z}(q) R_{z,y}(q) = 0.$$

In particular, if $\ell(x, y)$ is even, we have

$$R_{x,y}(q) = \frac{1}{2} \sum_{\substack{x<z<y \\ \ell(x,z) \text{ odd}}} R_{x,z}(q)R_{z,y}(q) - \frac{1}{2} \sum_{\substack{x<z<y \\ \ell(x,z) \text{ even}}} R_{x,z}(q)R_{z,y}(q) \tag{1}$$

Note that equation (1) allows to compute the R-polynomial associated with an interval $[x, y]$ of even length ℓ, once the R-polynomials associated with the subintervals of $[x, y]$ of length $< \ell$ are known. In particular, if the combinatorial invariance of the R-polynomials is true for intervals of length $< \ell$, then it is true also for intervals of length ℓ.

In order to give a combinatorial interpretation of the R-polynomials, another family of polynomials, known as the *$\widetilde{R}$-polynomials*, has been introduced.

Proposition 2.4. *Let $x, y \in W$. Then there is a unique polynomial $\widetilde{R}_{x,y}(q) \in \mathbb{N}[q]$ such that*

$$R_{x,y}(q) = q^{\frac{\ell(x,y)}{2}} \widetilde{R}_{x,y}\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right).$$

The advantage of the $\widetilde{R}$-polynomials over the R-polynomials is that they have non-negative integer coefficients. In fact there is a nice combinatorial interpretation for them. In order to give that, we introduce some notation.

Given $x, y \in W$, with $x < y$, we denote by $Paths(x, y)$ the set of paths in BG from x to y. The *length* of $\Delta = (x_0, x_1, \dots, x_k) \in Paths(x, y)$, denoted by $|\Delta|$, is the number k of its edges. Now let $\prec$ be a fixed *reflection ordering* on the set T of reflections (see, e.g., [1, §5.2] for the definition and for a proof of existence). A path $\Delta = (x_0, x_1, \dots, x_k) \in Paths(x, y)$, with

$$x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} x_k,$$

is said to be *increasing* with respect to the order $\prec$ if $t_1 \prec t_2 \prec \cdots \prec t_k$. Denote by $Paths^{\prec}(x, y)$ the set of all paths in $Paths(x, y)$ which are increasing with respect to $\prec$. The following result is due to Dyer [5].

Theorem 2.5. *Let W be a Coxeter and let $x, y \in W$, with $x < y$. Set $\ell = \ell(x, y)$. Then*

$$\widetilde{R}_{x,y}(q) = \sum_{k=1}^{\ell} c_k q^k, \qquad (2)$$

where

$$c_k = |\{\Delta \in Paths^{\prec}(x, y) : |\Delta| = k\}|, \quad \text{for every } k \in [\ell].$$

Equation (2) can be refined. Every path from x to y in BG necessarily has a length of the same parity of ℓ, so $c_k = 0$ if $k \not\equiv \ell\,(2)$. Furthermore, by the EL-shellability of the Bruhat order (see, e.g., [5]), in BG there is exactly one increasing path from x to y of length ℓ, thus $c_\ell = 1$.

Finally, let us introduce the notion of absolute length of a pair. We recall that the *absolute length* of $x \in W$, denoted by $a\ell(x)$, is defined as the minimum k such that x can be written as the product of k reflections. By [6, Theorem 1.2], $a\ell(x)$ is the (oriented) distance between e and x in BG. And by [6, Theorem 2.3], the coefficient of q^k in $\widetilde{R}_{x,y}(q)$ is non-zero if and only if there is a path in BG of length k. These facts suggest the following definition.

Definition 2.6. *Let $x, y \in W$, with $x < y$. The* absolute length *of (x, y), denoted by $a\ell(x, y)$, is the (oriented) distance between x and y in BG.*

By the results in [6], this notion generalizes that of absolute length of an element. In fact, given $x \in W$, we have $a\ell(x) = a\ell(e, x)$. On the other hand, by [2, Proposition 6.1], there exists $m \equiv \ell(x, y)\ (2)$ such that the coefficient of q^k in $\widetilde{R}_{x,y}(q)$ is non-zero if and only if $k \in [m, \ell(x, y)]$. It turns out that this m is exactly $a\ell(x, y)$. Putting all together, we have the following.

Corollary 2.7. *Let W be a Coxeter and let $x, y \in W$, with $x < y$. Set $\ell = \ell(x, y)$ and $a\ell = a\ell(x, y)$. Then*

$$\widetilde{R}_{x,y}(q) = q^{\ell} + c_{\ell-2}\, q^{\ell-2} + \cdots + c_{a\ell+2}\, q^{a\ell+2} + c_{a\ell}\, q^{a\ell}.$$

where

$$c_k = |\{\Delta \in Paths^{\prec}(x, y) : |\Delta| = k\}| \geq 1$$

for every $k \in [a\ell, \ell - 2]$, with $k \equiv \ell$ (2).

It is worth noting that the relation $x \to y$ is determined by the poset structure of $[x, y]$ (see [4]). Combining that with the results in [6], we have the following.

Proposition 2.8. *Let W be a Coxeter group and let $x, y \in W$, with $x < y$. The absolute length $a\ell(x, y)$ is a combinatorial invariant, that is, it depends only on the poset structure of $[x, y]$.*

The following is the well-known *combinatorial invariance conjecture*, which was stated by Lusztig and independently by Dyer in the 1980s, and since then it has remained unsolved.

Conjecture 2.9. *Let W_1 and W_2 be two Coxeter groups and let $x, y \in W_1$, with $x < y$, and $u, v \in W_2$, with $u < v$. Then*

$$[x, y] \cong [u, v] \quad \Rightarrow \quad P_{x,y}(q) = P_{u,v}(q).$$

In other words, the Kazhdan-Lusztig polynomial associated with (x, y) supposedly depends only on the poset structure of $[x, y]$. The combinatorial invariance conjecture is equivalent to the analogous statement for the R-polynomials, by Theorem 2.2, and for the $\widetilde{R}$-polynomials, by Proposition 2.4. It is known to be true if $[x, y]$ is a lattice (see [2, Theorem 6.3]) and holds for intervals up to length 4 (see, e.g., [1]). In this paper we prove that this is true for intervals of the symmetric group of length 5 and 6.

3. Main result

As usual, we denote by S_n the *symmetric group* over n elements, that is the set of all bijections of $[n]$ onto itself, and call its elements *permutations*. The symmetric group S_n is known to be a Coxeter group, with generators given by the simple transpositions $(i, i+1)$, for $i \in [n-1]$.

Given a poset P and $x, y \in P$, with $x < y$, we denote by $a(x, y)$ and $c(x, y)$, respectively, the number of *atoms* and *coatoms* of the interval $[x, y]$, that is

$$a(x, y) = |\{z \in [x, y] : x \lhd z\}| \quad \text{and} \quad c(x, y) = |\{z \in [x, y] : z \lhd y\}|,$$

and by $cap(x, y)$ its *capacity*, that is

$$cap(x, y) = \min\{a(x, y), c(x, y)\}.$$

We also denote by $\mathcal{B}_k$ the *boolean algebra* of rank k, that is the family $\mathcal{P}([k])$ of all subsets of $[k]$ partially ordered by inclusion. The following is the main result of this paper.

Theorem 3.1. *Let $x, y \in S_n$, for some n, with $x < y$ and $\ell(x, y) = 5$. Set $a = a(x, y)$, $c = c(c, y)$ and $cap = cap(x, y)$. Then*

$$\widetilde{R}_{x,y}(q) = \begin{cases} q^5 + 2q^3 + q, & \textit{if} \quad \{a, c\} = \{3, 4\}, \\ q^5 + 2q^3, & \textit{if} \quad a = c = 3, \\ q^5 + q^3, & \textit{if} \quad cap \in \{4, 5\} \textit{ but } [x, y] \ncong \mathcal{B}_5, \\ q^5, & \textit{if} \quad cap \in \{6, 7\} \textit{ or } [x, y] \cong \mathcal{B}_5. \end{cases}$$

A sketch of the proof of Theorem 3.1 will be given in §8. As a consequence of it, we have the combinatorial invariance of Kazhdan-Lusztig polynomials for intervals of length 5 and 6 intervals in the symmetric group, as we state in the following.

Corollary 3.2. *Let $x, y \in S_n$, with $x < y$, and $u, v \in S_m$, with $u < v$, for some n and m, be such that $\ell(x, y) = \ell(u, v) \in \{5, 6\}$. Then*

$$[x, y] \cong [u, v] \quad \Rightarrow \quad P_{x,y}(q) = P_{u,v}(q).$$

Proof. Theorem 3.1 and Proposition 2.4 imply the combinatorial invariance of the R-polynomials for intervals of length 5. For intervals of length 6, it follows from equation (1). Finally, by Theorem 2.2, the assertion for the Kazhdan-Lusztig polynomials follows. □

4. Diagram of a pair of permutations

To denote a permutation $x \in S_n$ we often use the *one-line notation*: we write $x = x_1 x_2 \dots x_n$, to mean that $x(i) = x_i$ for every $i \in [n]$. The *diagram* of a permutation $x \in S_n$ is the subset of $\mathbf{N}^2$ so defined:

$$Diag(x) = \{(i, x(i)) : i \in [n]\}$$

The symmetric group is a Coxeter group and there is a nice combinatorial characterization of the Bruhat order relation in it. In order to give that, we introduce the following notation: for $x \in S_n$ and $(h, k) \in [n]^2$, we set

$$x[h, k] = |\{i \in [h] : x(i) \in [k, n]\}, \tag{3}$$

that is, $x[h,k]$ is the number of points of the diagram of x lying in the upper-left quarter plane with origin at (h,k). And given $x, y \in S_n$ and $(h,k) \in [n]^2$, we set

$$(x,y)[h,k] = y(h,k) - x(h,k). \tag{4}$$

The characterization is the following.

Theorem 4.1. *Let $x, y \in S_n$. Then*

$$x \le y \quad \Leftrightarrow \quad (x,y)[h,k] \ge 0, \quad \textit{for every } (h,k) \in [n]^2.$$

It is useful to extend the notation introduced in (3) and (4) to every $(h,k) \in \mathbf{R}^2$. We call the mapping $(h,k) \mapsto (x,y)[h,k]$, which associates with every $(h,k) \in \mathbf{R}^2$ the integer $(x,y)[h,k]$, the *multiplicity mapping* of the pair (x,y). Theorem 4.1 can be reformulated as follows:

$$x \le y \quad \Leftrightarrow \quad (x,y)[h,k] \ge 0, \quad \textit{for every } (h,k) \in \mathbf{R}^2.$$

Definition 4.2. *Let $x, y \in S_n$. The* diagram *of the pair (x,y) is the collection of:*

(1) *the diagram of x;*
(2) *the diagram of y;*
(3) *the multiplicity mapping $(h,k) \mapsto (x,y)[h,k]$.*

We pictorially represent the diagram of a pair (x,y) with the following convention: the diagram of x (respectively, y) is denoted by black dots (respectively, white dots) and, if $x < y$, then the mapping $(h,k) \mapsto (x,y)[h,k]$ is represented by colouring the preimages of different positive integers with different levels of grey, with the rule that with a lower integer corresponds a lighter grey. For example, the diagram of the pair $(315472986, 782496315)$ is depicted in Figure 1.

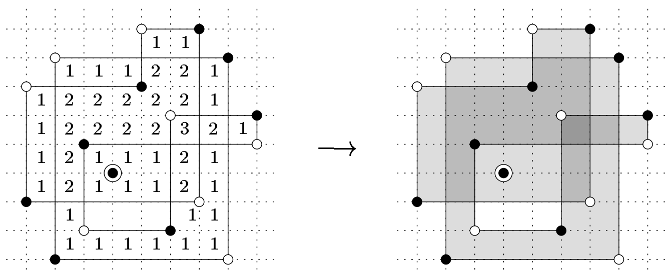


FIGURE 1. Diagram of a pair of permutations.

Finally, it is useful to give the following definition.

Definition 4.3. *Let $x, y \in S_n$, with $x < y$. The* support *of the pair (x,y) is*

$$\Omega(x,y) = \{(h,k) \in \mathbf{R}^2 : (x,y)[h,k] > 0\}.$$

5. FROM THE DIAGRAM TO THE POSET STRUCTURE

In this section we show how it is possible, starting from the diagram of a pair (x,y), with $x < y$, to get information about the poset structure of the interval $[x,y]$.

5.1. **Symmetries.** In general Coxeter groups, it is known that the mapping $x \mapsto x^{-1}$ is an isomorphism of the Bruhat order. Also, finite Coxeter groups always have a maximum, usually denoted w_0, and the mappings $x \mapsto xw_0$ and $x \mapsto w_0x$ are anti-isomorphisms. It follows that $x \mapsto w_0xw_0$ is an isomorphism.

In the symmetric group these facts can be described in a nice pictorial way. In fact the maximum of S_n is $w_0 = n\, n-1 \ldots 2\, 1$, and given $x \in S_n$, the diagrams of x^{-1}, xw_0, w_0x can be respectively obtained from the diagram of x by a reflection with respect to the lines $\{(h,k) \in \mathbf{R}^2 : h = k\}$, $\{(h,k) \in \mathbf{R}^2 : h = (n+1)/2\}$ and $\{(h,k) \in \mathbf{R}^2 : k = (n+1)/2\}$. The diagram of w_0xw_0 can be obtained from that of x by a reflection with respect to the point $((n+1)/2, (n+1)/2)$.

So, given $x, y \in S_n$, with $x < y$, there are four trivial associated intervals belonging to the same isomorphism class, and four belonging to the dual class. This means that, in order to study all possible isomorphism types that can occur in the symmetric group, it is enough to consider diagrams of pairs of permutations up to these symmetries.

5.2. **Atoms and coatoms.** A combinatorial characterization of the covering relation in the Bruhat order of the symmetric group is given in terms of free rises. We recall that, given $x \in S_n$, a *free rise* of x is a pair (i, j), with $i < j$ and $x(i) < x(j)$, such that there is no $k \in \mathbf{N}$, with $i < k < j$ and $x(i) < x(k) < x(j)$. It is known that, given $x, y \in S_n$, then $x \lhd y$ if and only if $y = x(i, j)$, where (i, j) is a free rise of x.

If (i, j) is a free rise of x, then the *rectangle associated* with (i, j) is:

$$Rect_x(i,j) = \{(h,k) \in \mathbf{R}^2 : i \leq h < j \,,\, x(i) < k \leq x(j)\}.$$

Now let $x, y \in S_n$, with $x < y$. In order to describe the atoms of the interval $[x, y]$, we say that a free rise (i, j) of x is *good* with respect to y if

$$Rect_x(i,j) \subseteq \Omega(x,y).$$

The following gives a characterization of the atoms of an interval.

Proposition 5.1. *Let $x, y \in S_n$, with $x < y$. Then z is an atom of $[x, y]$ if and only if $z = x(i, j)$, where (i, j) is a free rise of x good with respect to y.*

Proof. Let (i, j) be a free rise of x, and let $z = x(i, j)$. We know that $x \lhd z$. Thus z is an atom of $[x, y]$ if and only if $z \leq y$. As it can be easily checked, for every $(h, k) \in \mathbf{R}^2$, we have

$$(z,y)[h,k] = \begin{cases} (x,y)[h,k] - 1, & \text{if } (h,k) \in Rect_x(i,j), \\ (x,y)[h,k], & \text{otherwise.} \end{cases}$$

Then, by Theorem 4.1, $z \leq y$ if and only if $(x, y)[h, k] \geq 1$ for every $(h, k) \in Rect_x(i, j)$, that is if and only if $Rect_x(i, j) \subseteq \Omega(x, y)$. □

In a specular way, we can define a *free inversion* of y, as a pair (i, j), with $i < j$ and $y(i) > y(j)$, such that there is no $k \in \mathbf{N}$, with $i < k < j$ and $y(i) > y(k) > y(j)$. Note that $x \lhd y$ if and only if $x = y(i, j)$, where (i, j) is a free inversion of y. The *rectangle associated* with (i, j) is:

$$Rect_y(i,j) = \{(h,k) \in \mathbf{R}^2 : i \leq h < j \,,\, y(j) < k \leq y(i)\}.$$

Given $x, y \in S_n$, with $x < y$, we say that a free inversion (i, j) of y is *good* with respect to x if

$$Rect_y(i,j) \subseteq \Omega(x,y).$$

Next result gives a characterization of the coatoms of an interval, and its proof is completely specular to that of Proposition 5.1.

Proposition 5.2. *Let $x, y \in S_n$, with $x < y$. Then w is a coatom of $[x, y]$ if and only if $w = y(i, j)$, where (i, j) is a free inversion of y good with respect to x.*

Going back to the example shown in Figure 1, the free rises of x good with respect to y, and the free inversions of y good with respect to x are illustrated in Figure 2. We can conclude that

$$a(x,y) = c(x,y) = 11.$$

6. From the diagram to the $\widetilde{R}$-polynomial

By Theorem 2.5, computing an $\widetilde{R}$-polynomial means computing the number of increasing chains in BG with respect to a given reflection ordering on the reflections. In this section we show how it can be done for an interval $[x, y]$ of the symmetric group, in terms of the diagram of (x, y).

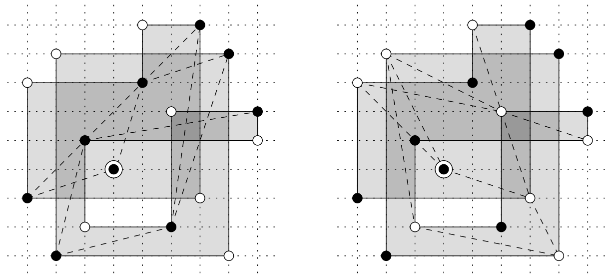


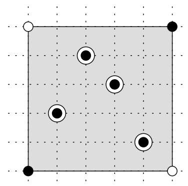
FIGURE 2. Atoms and coatoms.

6.1. **Symmetries.** Given a Coxeter group W, and given $x, y \in W$, with $x < y$, it is known (see, e.g., [1]) that $\widetilde{R}_{x,y}(q) = \widetilde{R}_{x^{-1},y^{-1}}(q)$ and, if W is finite, $\widetilde{R}_{x,y}(q) = \widetilde{R}_{yw_0,xw_0}(q) = \widetilde{R}_{w_0y,w_0x}(q) = \widetilde{R}_{w_0xw_0,w_0yw_0}(q)$. In the symmetric group this means that the eight pairs obtained from (x, y) by the symmetries previously described, have all the same $\widetilde{R}$-polynomial associated. Thus, in order to list all possible $\widetilde{R}$-polynomials that can occur in the symmetric group, it is enough to consider diagrams up to those symmetries.

6.2. **The *stair method*.** It is known that in the symmetric group S_n the reflections are the transpositions ($T = \{(i,j) : i,j \in [n]\}$) and that a possible reflection ordering on them is the lexicographic order (see, e.g., [1]). From now on we always assume this order fixed on the transpositions. For instance, in S_4:

$$(1,2) \prec (1,3) \prec (1,4) \prec (2,3) \prec (2,4) \prec (3,4).$$

Let $x, y \in S_n$, with $x < y$. We start considering the case in which (x, y) is an edge of BG, that is $y = x(i,j)$ for some transposition (i, j). In this case the support is a rectangle. A possible diagram of (x, y) is, for example, the following:



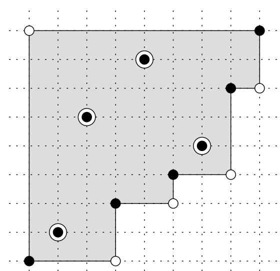
Now let (x, y) be not necessarily an edge of BG. There is a nice way to describe the increasing paths in BG from x to y. Suppose given such a path:

$$x = x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} x_k = y. \tag{5}$$

We recall that the *difference index* of x with respect to y, denoted by di, is the minimal k such that $x(k) \neq y(k)$. We also call $x^{-1}y(di)$ the *stair index* of x with respect to y, and denote it by si. Note that, by definition, $di < si$. Also note that in (5) necessarily $t_1 = (di, j_1)$, for some j_1, otherwise the path would not be increasing. We now consider the case $t_h = (di, j_h)$, for every $h \in [k]$, with $j_k = si$:

$$x = x_0 \overset{(di,j_1)}{\longrightarrow} x_1 \overset{(di,j_2)}{\longrightarrow} \cdots \overset{(di,j_{k-1})}{\longrightarrow} x_{k-1} \overset{(di,si)}{\longrightarrow} x_k = y.$$

We call such a path a *stair path*, because of the shape of the support: it is a stair with k steps. Here is an example:



In the general case, the increasing path in (5) is a sequence of stair paths. It is useful to give the following definitions, which formalize the notion of stair.

Definition 6.1. *Let $x \in S_n$. A* stair *of x is a sequence $s = (j_0, j_1, \dots, j_k) \in [n]^k$, which is increasing and such that $(x(j_0), x(j_1), \dots, x(j_k))$ is also increasing. The* stair area *associated with s is the subset of $\mathbf{R}^2$ so defined:*

$$Stair_x(s) = \bigcup_{i \in [k]} \{(a, b) \in \mathbf{R}^2 : j_{i-1} \leq a < j_i \, , \, x(j_{i-1}) < b \leq x(j_i)\}.$$

We also say that the permutation $x(j_0, j_k, \dots, j_1)$ is obtained *from x by* performing *the stair s.*

Definition 6.2. *Let $x, y \in S_n$, with $x < y$. A stair s of x is said to be* good *with respect to y if*

$$Stair_x(s) \subseteq \Omega(x, y)$$

Then we have the following, whose proof is similar to that of Proposition 5.1.

Proposition 6.3. *Let $x, y \in S_n$, with $x < y$. Let $s = (j_0, j_1, \dots, j_k)$ be a stair of x. Then $x(j_0, j_k, \dots, j_1) \leq y$ if and only if s is good with respect to y.*

An increasing path from x to y is obtained by performing subsequent stairs, choosing at each step one of the leftmost good stairs. Next definition gives the last tool we need.

Definition 6.4. *Let $x, y \in S_n$, with $x < y$. An* initial stair *of (x, y) is a stair s of x good with respect to y which starts with di and ends with si:*

$$s = (di, j_1, j_2, \dots, j_{k-1}, si).$$

Examples of initial stairs will be shown in Figure 3. Note that an initial stair of (x, y) always exists. One can be obtained, for example, by choosing its indices as small as possible.

We can finally give a general algorithm, which allows to generate all possible increasing paths in BG from x to y, and thus to compute the coefficients of the $\widetilde{R}$-polynomial. We call it the *stair method.* Starting from the diagram of (x, y), all the increasing paths from x to y can be recursively constructed as follows:

(1) choose an initial stair of (x, y);
(2) call x_1 the permutation obtained from x by performing the choosen stair (note that $x_1 \leq y$, by Proposition 6.3);
(3) recursively apply the procedure on (x_1, y).

An example is shown in Figure 3. Here we take the example introduced in Figure 1, and apply the algorithm to that, making some choices. The corresponding increasing path in BG which arises has length 9 and is:

$$x \xrightarrow{(1,4)} \bullet \xrightarrow{(1,5)} x_1 \xrightarrow{(2,3)} \bullet \xrightarrow{(2,8)} x_2 \xrightarrow{(3,6)} x_3 \xrightarrow{(4,5)} x_4 \xrightarrow{(5,7)} x_5 \xrightarrow{(6,8)} \bullet \xrightarrow{(6,9)} y.$$

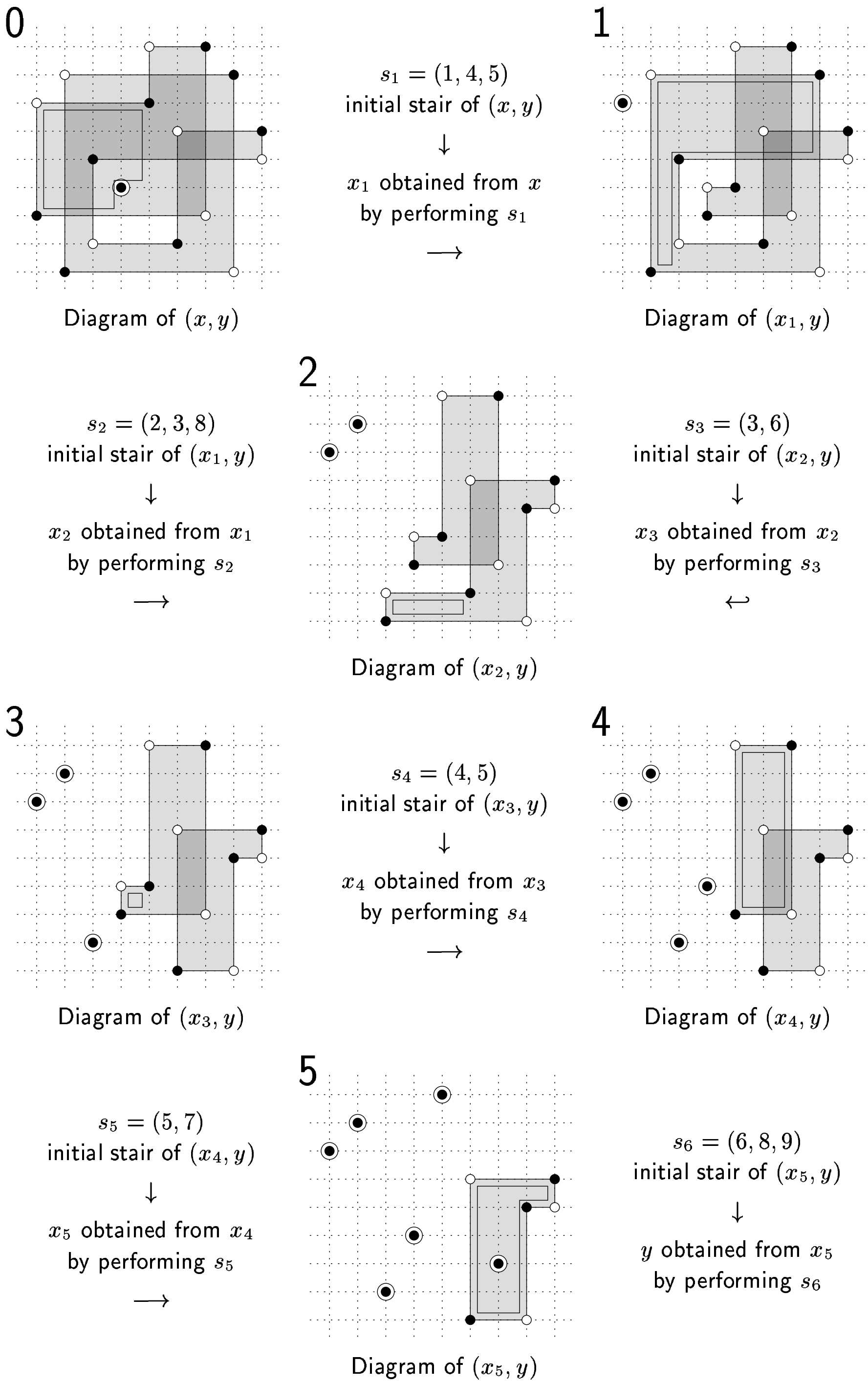


FIGURE 3. The stair method.

We could have choosen, for example, the initial stairs $(1, 5)$ or $(1, 3, 5)$ of (x, y), instead of $(1, 4, 5)$, or the initial stair $(6, 7, 8, 9)$ of (x_5, y), instead of $(6, 8, 9)$. Considering all possible choices, we can obtain all increasing paths from x to y, and it turns out that they are 10 and that:

$$\widetilde{R}_{x,y}(q) = q^{13} + 4q^{11} + 4q^9 + q^7.$$

Note that the only increasing path in BG from x to y of length $\ell(x, y)$, whose existence and uniqueness is guaranteed by the EL-shellability, is obtained by choosing in the algorithm always the lexicographically minimal stair.

6.3. **Special cases.** The stair method allows to compute the $\widetilde{R}$-polynomial for some general classes of pairs (x, y).

Definition 6.5. *Let $x, y \in S_n$, with $x < y$. We say that*

(1) (x, y) *has the* 01-multiplicity property *if*

$$(x,y)[h,k] \in \{0,1\} \quad \textit{for every } (h,k) \in \mathbf{R}^2;$$

(2) (x, y) *is* simple *if it has the 01-multiplicity property and* $\{i \in [n] : x(i) = y(i)\} = \emptyset$;

(3) (x, y) *is a* permutaomino *if it is simple and* $\Omega(x, y)$ *is connected.*

Examples are shown in Figure 4.

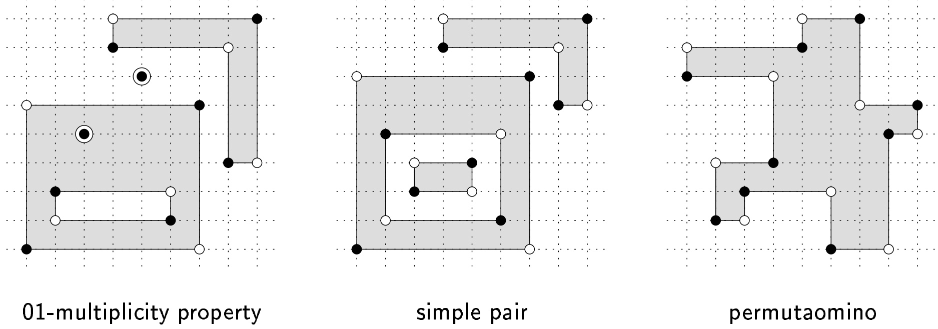


FIGURE 4. Special pairs of permutations.

Definition 6.6. *Let $x, y \in S_n$, with $x < y$, be such that (x, y) satisfies the 01-multiplicity property. Then the* fixed point multiplicity *of (x, y) is*

$$fpm(x,y) = |\{i \in [n] : x(i) = y(i),\ (x,y)[i, x(i)] = 1\}|.$$

Here we have the expressions of the $\widetilde{R}$-polynomial for these classes of pairs.

Proposition 6.7. *Let $x, y \in S_n$, with $x < y$.*

(1) *If (x, y) has the 01-multiplicity property, then*

$$\widetilde{R}_{x,y}(q) = (q^2+1)^{fpm(x,y)} q^{a\ell(x,y)},$$

(2) *If (x, y) is simple then*

$$\widetilde{R}_{x,y}(q) = q^{\ell(x,y)},$$

(3) *If (x, y) is a permutaomino then*

$$\widetilde{R}_{x,y}(q) = q^{n-1},$$

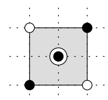
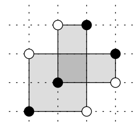
A particular case in which (x, y) has the 01-multiplicity property, is when (x, y) is an edge of BG. Then we get the following.

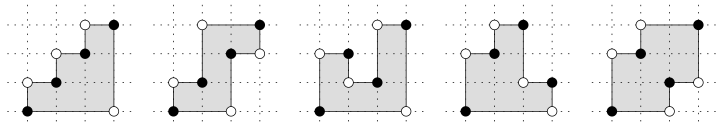
Proposition 6.8. *Let $x, y \in S_n$, with $x \to y$ and $\ell(x, y) = 2p + 1$. Then*

$$\widetilde{R}_{x,y} = q(q^2+1)^p.$$

7. Intervals of length ≤ 5

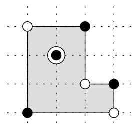
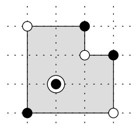
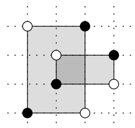
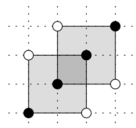
7.1. **Length 3.** The diagrams of pairs (x, y), with $\ell(x, y) = 3$, which are not trivially reducible, are, up to symmetries, exactly the following:

the *eye* diagram:  the *rabbit* diagram: 

permutaominos with 8 edges: 

It is known that only the so called k-crowns can occur as intervals of length 3 in Coxeter groups, and in particular in the symmetric group there are only 2, 3 and 4-crowns. Namely, the eye diagram corresponds to a 2-crown, and the last permutaomino to a 4-crown. The other permutaominoes and the rabbit diagram correspond to a 3-crown. Finally, $\widetilde{R}_{x,y}(q) = q^3 + q$, for the eye diagram, and q^3 in all other cases.

7.2. **Length 4.** Among the diagrams corresponding to length 4 intervals, we mention all the permutaominos with 10 edges. And the following four:

1.  2.  3.  4. 

We call them the *essential diagrams* of length 4. They are, up to symmetries, the only ones obtained by "enlarging" the eye diagram in S_4, which are not trivially reducible.

For general length 4 intervals, as poset types we have the products of the 2, 3 and 4-crown times $\{0, 1\}$, and two irreducible ones. And the $\widetilde{R}$-polynomial can be either $q^4 + q^2$ or q^2.

7.3. **Length 5.** For completeness, even if we don't use it in the proof of our main result, we mention that in [7] all the intervals of length 5 occurring in the symmetric group have been listed. They have been generated using a Maple package by J. R. Stembridge.

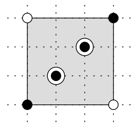
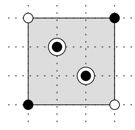
8. Sketch of proof of the main result

Sketch of proof of Theorem 3.1. Suppose known the poset structure of $[x, y]$, with $\ell(x, y) = 5$. We want show that it allows to determine the polynomial $\widetilde{R}_{x,y}(q)$.

By Proposition 2.8, the poset structure of $[x, y]$ determines $a\ell(x, y) \in \{1, 3, 5\}$.

If $a\ell(x, y) = 5$, then $\widetilde{R}_{x,y}(q) = q^5$ is determined. Note that in this case the poset $[x, y]$ is a lattice and it is known that this implies either $cap(x, y) \geq 6$, or $[x, y] \cong \mathcal{B}_5$.

If $a\ell(x, y) = 1$, then (x, y) is an edge of BG. So the diagram of (x, y) is one of the following:

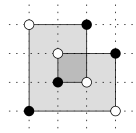
 

In this case, we have $\{a(x, y), c(x, y)\} = \{3, 4\}$ and, by Proposition 6.8

$$\widetilde{R}_{x,y}(q) = q^5 + 2q^3 + q.$$

Finally suppose $a\ell(x,y) = 3$. In this case $\widetilde{R} = q^5 + bq^3$, for some $b \in \mathbf{N}$. The only diagrams in S_4 of length 5 are, up to symmetries, exactly two: the one corresponding to the edges of BG, already considered, and the following:

the *heart* diagram:

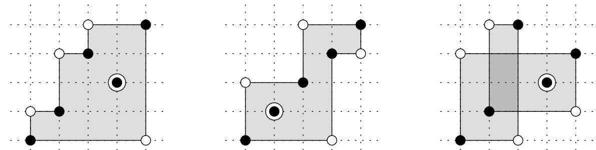


In this case $a(x,y) = c(x,y) = 3$, and applying the stair method we obtain

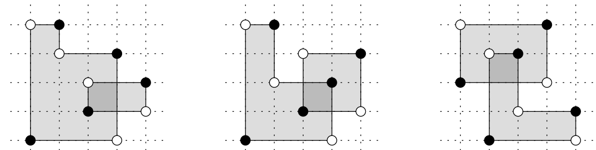
$$\widetilde{R}_{x,y}(q) = q^5 + 2q^3.$$

All other cases can be obtained by "enlarging" the essential diagrams of length 4 in S_5. Considering only non trivially reducible cases, it turns out that two general situations can occur:

(1) the diagram of (x,y) is obtained from one of the diagrams of length 3 corresponding to a 3-crown, by adding one fixed point with multiplicity 1. Here are some examples:



(2) the diagram of (x,y) is the overlapping of a permutaomino with 4 edges and one with 6 edges. A few examples:



After a patient enumeration of all possible cases, and using the interpretation of atoms and coatoms in terms of the diagram, it turns out that in all these cases $cap(x,y) \in \{4,5\}$, but the boolean algebra $\mathcal{B}_5$ never occurs. And applying the stair method, we get

$$\widetilde{R}_{x,y}(q) = q^5 + q^3.$$

Putting all together, we get our result. □

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Institut Mittag-Leffler, Auravägen 17, 18260, Djursholm, Sweden
E-mail address: `incitti@math.kth.se`