LOCAL ACTION OF THE SYMMETRIC GROUP AND GENERALIZATIONS OF QUASI-SYMMETRIC FUNCTIONS

EXTENDED ABSTRACT

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Abstract. This paper is a tentative to better understand the result of Garsia and Wallach showing that the ring of quasi-symmetric functions is free as a module over the ring of symmetric functions. We conjecture that their result is a particular case of a more general situation whose construction is decribed here.

We consider a certain class of actions of the symmetric group $\mathfrak{S}_n$ on polynomials $\mathbb{K}[X_n] = \mathbb{K}[x_1, \ldots, x_n]$ called *local actions*, which are very similar to the one of [6, 7]. After classifying these actions, we study the sub-class of these actions whose set of fixed polynomial is a sub-algebra of $\mathbb{K}[X_n]$. This gives rise to an infinite hierarchy of sub-Hopf-algebras of QSym, interpolating between QSym and Sym. We conjecture that these algebra are free modules over Sym as is suggested by an explicit formula giving the Hilbert series of the quotient.

Contents

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1. Introduction

The symmetric polynomials Sym are by definition the polynomials of $\mathbb{K}[X_n] := \mathbb{K}[x_1, \ldots, x_n]$ which are invariant under the action of the symmetric group $\mathfrak{S}_n$ on polynomials. They form a sub-algebra Sym := $\mathbb{K}[X_n]^{\mathfrak{S}_n}$ of $\mathbb{K}[X_n]$, and consequently, polynomials can be considered as a module over Sym. It is well known that this module is free of rank $n!$ and this result is now widely interpreted for example in the cohomology or the Grothendieck ring of the flag manifold.

In his study of descents of permutations, Gessel defined a generalization of the notion of symmetric functions namely the quasi-symmetric functions QSym [4]. This algebra as been extensively studied: it can also be endowed with a natural Hopf algebra structure which is dual to the Hopf algebra **NCSF** of noncommutative symmetric functions [11, 3]. Quasi-symmetric functions are defined as the set of polynomials with a certain partial symmetry property. However, in [6, 7], the author described a new action of the symmetric group and its Hecke algebra on the space of polynomials whose invariant are exactly the quasi-symmetric polynomials. At this point, it is important to realize that the action is not compatible with the product, so that there is no way to deduce from it that quasi-symmetric polynomials form an algebra.

A further step in the study of quasi-symmetric polynomials is the result of Garsia and Wallach [2] showing that QSym is free as a module on Sym. This was conjectured by F. Bergeron and C. Reutenauer and independently by J.-Y. Thibon when he gave me my thesis subject ! Unfortunately I was not able to solve it.

As in the case of polynomials the module of QSym over Sym is of rank $n!$, which can be easily deduced from the generating series. However the proof of Garsia and Wallach relies on a very subtle analyze of the generating series and do not gives no good explanation of the link between the $n!$ and the symmetric group. The present paper is actually a tentative to understand their result. Though there is apparently no link between the rank $n!$ of the module and the existence of the action of the symmetric group, both of these properties seem to generalize. This give a new light, which allows us to give a combinatorial interpretation of the Hilbert series of the module, assuming its freeness:

$$\text{Hilb}_t(\text{QSym}^r(X_n)) = \sum_{\sigma\in\mathfrak{S}_n} \frac{t^{\text{Maj}(\sigma)+r(n-\text{Fix}(\sigma))}}{(1-t)(1-t^2)\cdots(1-t^n)}\,, \tag{1}$$

where Maj is the major index ant Fix is the number of fixed points. Such a result is quite unexpected even in the case of $QSym$.

The paper is structured as follows. In a first part, we study a class of actions of the symmetric group on polynomials generalizing those of [6, 7]. In these actions, the elementary transposition σ_i acts only on the variables x_i and x_{i+1}, and therefore we call these action *local.* Then, we investigate the case where the set of fixed points under one of these action form a sub-algebra of the algebra of polynomials. This gives rise to an infinite hierarchy of sub-algebras of QSym, interpolating between QSym and Sym. We further show that, when the number of variables is infinite, these algebras are actually sub-Hopf-algebras of QSym and describe the dual Hopf algebras as quotient of **NCSF**. Finally, we show that, as in the case of QSym these algebras are free, and that, in the finite number of variable case, they seem to be Cohen-Macaulay as indicated by their generating series. It seems that Garsia and Wallach's proof could be adapted to this general case, and further work is in progress in this direction.

2. Background

2.1. **Quasi-symmetric functions.** Let $X := \{x_1 < x_2 < \cdots < x_n\}$ denote a totally ordered set of commutative indeterminates. X is called the *alphabet*. By $\mathcal{P}(X)$ (resp. $\mathcal{P}_k(X)$), we mean the set of the subsets (resp. k-elements subsets) of the alphabet X.

Let m be the monomial $x_1^{m_1} \dots x_n^{m_n}$ where the m_i's are possibly zero. For readability, we identify m with the integer vector $[m_1, m_2, \dots, m_n]$. We define the *support* of m as the set $A \in \mathcal{P}(X)$ of the x_i's whose exponent are non-zero, as well as the composition I obtained by removing the zeros in the sequence $(m_1, m_2, \dots, m_n)$. In the sequel we write A^I in place of the monomial m. For example if $X = \{x_1 < x_2 < x_3 < x_4\}$, we write $x_1^2 x_3 = [2, 0, 1, 0] = \{x_1, x_3\}^{(2,1)}$ and $x_1^3 x_2^5 x_4 = [3, 5, 0, 1] = \{x_1, x_2, x_4\}^{(3,5,1)}$.

A polynomial $f \in \mathbb{C}[X]$ is said to be *quasi-symmetric* if and only if for each composition $I = (i_1, \dots, i_r)$ the coefficient of the monomial

$$x_{j_1}^{i_1} x_{j_2}^{i_2} \dots x_{j_r}^{i_r} \tag{2}$$

is independent of the choice of $j_1 < j_2 < \cdots < j_r$. With our support and exponent notations it says that the coefficient of A^I is independent of the set of variables $A \in \mathcal{P}_r(X)$.

The quasi-symmetric polynomials form a subalgebra of $\mathbb{C}[X]$ denoted by QSym_n. It is often convenient to let n tends toward ∞ and to take the inverse limit in the category of graded ring. Then, we get an algebra called the algebra of *quasi-symmetric functions* [4]. Such functions can be seen as formal sums of monomials on an infinite alphabet $X := \{x_1 < x_2 < \cdots < x_n < \cdots\}$.

It is clear that the family of so-called *quasi-monomial functions* defined by

$$M_I := \sum_{A \in \mathcal{P}_r(X)} A^I = \sum_{j_1 < \cdots < j_r} x_{j_1}^{i_1} \cdots x_{j_r}^{i_r} = \sum_{k \to I} x^k \tag{3}$$

labeled by compositions $I := (i_1 \dots, i_r)$ form a basis of QSym, and the last sum is extended to all integer vectors $k \to I$ of length n obtained by inserting zeros in the composition I. For example,

$$M_{(2,1)} = \{x_1, x_2\}^{(2,1)} + \{x_1, x_3\}^{(2,1)} + \{x_1, x_4\}^{(2,1)} + \{x_2, x_3\}^{(2,1)} + \{x_2, x_4\}^{(2,1)} + \{x_3, x_4\}^{(2,1)}$$

which can be written more concisely as:

$$M_{(2,1)} = [2,1,0,0] \ + \ [2,0,1,0] \ + \ [2,0,0,1] \ + \ [0,2,1,0] + [0,2,0,1] \ + \ [0,0,2,1]$$

instead of $M_{(2,1)} = x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_2^2x_3 + x_2^2x_4 + x_3^2x_4$.

2.2. **Noncommutative symmetric functions.** The algebra of *noncommutative symmetric functions* [3] is the free associative algebra $\mathbf{NCSF} = \mathbb{C}\langle S_1, S_2, \dots\rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* symmetric functions. For a composition $I = (i_1, i_2, \dots, i_r)$, one sets $S^I := S_{i_1} S_{i_2} \dots S_{i_r}$. The family (S^I) is a linear basis of **NCSF**. A useful realization can be obtained by taking an infinite alphabet $A = \{ a_1, a_2, \dots \}$ and defining its complete homogeneous symmetric functions by the generating function

$$\sum_{n\geq 0} t^n S_n(A) = (1-t\,a_1)^{-1} (1-t\,a_2)^{-1} (1-t\,a_3)^{-1} \dots \tag{4}$$

Then, $S_n(A)$ appears as the sum of all nondecreasing words of length n. Note that these functions are not symmetric in the usual sense. They are invariant for a more subtle action of the symmetric group of the alphabet due to Lascoux and Schützenberger [9], (see also [3]). The role of Schur functions is played by the noncommutative *ribbon Schur functions* R_I defined by

$$R_I := \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} S^J . \tag{5}$$

The family (R_I) forms a basis of **NCSF**. In the realization of **NCSF** given by Equation (4), R_I reduces to the sum of all words of shape I [3].

The duality pairing $\langle \cdot | \cdot \rangle$ between QSym and **NCSF** is defined by $\langle M_I | S^J \rangle = \delta_{IJ}$ or equivalently $\langle F_I | R_J \rangle = \delta_{IJ}$ (*cf.* [11, 3]). This duality can be interpreted as the canonical duality between the Grothendieck groups respectively associated with finite dimensional and projective modules over 0-Hecke algebras (*cf.* [1, 8]).

3. Local actions

3.1. **Definitions.**

Definition 1. A *local action* of $\mathfrak{S}_n$ on $\mathbb{K}[X]$ is an action of $\mathfrak{S}_n$ on $\mathbb{K}[X]$ *i.e.* a morphism $\rho : \mathfrak{S}_n \longrightarrow \mathrm{End}(\mathbb{K}[X])$ which satisfies the following conditions:

(1) the elementary transposition σ_i acts only on the variables x_i and x_{i+1}, the other variables behaving as scalars;
(2) the action of an elementary transposition σ_i on a monomial $x_i^a x_{i+1}^b$ is independent of i and, depending on the values of a and b, either exchanges the variables or leaves the monomial invariant: for all i

$$\sigma_i(x_i^a x_{i+1}^b) = \begin{cases} x_i^b x_{i+1}^a & \text{for certain values of } a \text{ and } b\,, \\ x_i^a x_{i+1}^b & \text{for the other values}\,. \end{cases}$$

Consequently, the action of σ_1 on all monomials in x_1, x_2 fully determines the operation ρ.

Example 1. Let $m := [m_1, m_2, \dots m_n]$ be a monomial. Let also

$$\rho(\sigma_i)(m) := \begin{cases} [\dots, m_i, m_{i+1}, \dots] & \text{if } m_i \equiv m_{i+1} \pmod 2, \\ [\dots, m_{i+1}, m_i, \dots] & \text{otherwise.} \end{cases}$$

The hypothesis (1) and (2) are verified. We will see in the sequel that this actually defines a local action of the symmetric group.

The following proposition is an immediate consequence of the definition:

Proposition 3.1. *Let ρ a local action of $\mathfrak{S}_n$ on $\mathbb{K}[x_1, \dots, x_n]$. Then the restriction of ρ to $\mathfrak{S}_{n-1}$ is a local action on $\mathbb{K}[x_1, \dots, x_{n-1}]$.*

Let us give an other characterization of local actions:

Definition 2. Let ρ be a map of $\mathfrak{S}_n$ to $\mathrm{End}\,\mathbb{K}[X]$ which satisfies conditions (1) and (2) of the definition 1. To ρ we associate the relation $\mathcal{R}_\rho$ on the integers defined by

$$u \,\mathcal{R}_\rho\, v \quad \text{iff} \quad \sigma_1[u, v] = [u, v]. \tag{6}$$

Note that we do not suppose that ρ is a morphism, consequently it does not surely define an action. The conditions (1) and (2) ensure that, for all i

$$\rho(\sigma_i)(m) = \begin{cases} [\dots, m_i, m_{i+1}, \dots] & \text{if } m_i \,\mathcal{R}_\rho\, m_{i+1}, \\ [\dots, m_{i+1}, m_i, \dots] & \text{otherwise.} \end{cases} \tag{7}$$

Conversely, if $\mathcal{R}$ is a reflexive relation on the set of integers, Equation (7) defines a map from $\mathfrak{S}_n$ to $\mathrm{End}\,\mathbb{K}[X]$ which satisfies both conditions (1) and (2).

Proposition 3.2. *Let ρ be a map from $\mathfrak{S}_n$ to $\mathrm{End}\,\mathbb{K}[X]$ which satisfies both conditions (1) and (2) of definition 1. Let $\mathcal{R}_\rho$ be the associated relation. The following properties are equivalent:*

(1) *ρ is a morphism from $\mathfrak{S}_n$ to* $\mathrm{End}(\mathbb{K}[X])$,
(2) *$\mathcal{R}_\rho$ is an equivalence relation.*

Example 2. Let us go back to the former example. There are two equivalence classes: E for even and O for odd. To the monomial

$$m = [\ 0,\ 2,\ 1,\ 4,\ 3,\ 5,\ 1,\ 1, 2\ ]$$

we associate the word

$$C(m) = [\,\mathrm{E},\ \mathrm{E},\ \mathrm{O},\ \mathrm{E},\ \mathrm{O},\mathrm{O},\ \mathrm{O},\ \mathrm{O},\mathrm{E}\,]$$

and the two sequences of exponents

$$E_{\mathrm{E}} = (0, 2, 4, 2) \quad \text{and} \quad E_{\mathrm{O}} = (1, 3, 5, 1, 1).$$

Let $\sigma = 743652198$. Then,

$$\sigma(C(m)) = [\,\mathrm{O},\ \mathrm{O},\ \mathrm{O},\ \mathrm{E},\ \mathrm{O},\ \mathrm{E},\ \mathrm{E},\ \mathrm{E},\ \mathrm{O}].$$

We finally get

$$\rho(\sigma)m = [\ 1,\ 3,\ 5,\ 0,\ 1,\ 2,\ 4,\ 2, 1\ ].$$

3.2. **Characteristic and generalized Temperley-Lieb algebras.** In the previous subsection, we defined a family of actions of the symmetric group $\mathfrak{S}_n$ on polynomials. In this section we describe more precisely these actions in terms of the representations theory. To describe a representation of $\mathfrak{S}_n$, rather than giving its character we prefer the equivalent but easier to handle Frobenius characteristic. We only describe briefly this tool and refer to [10] for notations and details.

Recall that the direct sum $\bigoplus_{n\geq 0} R(\mathfrak{S}_n)$ of Grothendieck rings of all symmetric groups is in natural isomorphism with the ring of symmetric functions, by the so-called Frobenius characteristic map *ch*. It sends the irreducible character χ^λ to the Schur function s_λ. The product of symmetric functions corresponds to induction from $\mathfrak{S}_n \times \mathfrak{S}_p$ to $\mathfrak{S}_{n+p}$. One can get the value of the character χ on the conjugacy class (cycle type) indexed by the partition $\mu = (\mu_1, \ldots, \mu_p)$ by the scalar product:

$$\chi\big(\mu\big) = \langle \mathrm{ch}(\chi) | p_\mu \rangle\,. \tag{8}$$

where p_μ is the product of power sum symmetric functions.

The r-actions are compatible with the usual grading of polynomial rings. Hence, one can define the graded characteristic ch_t of the representation on $\mathbb{C}[X]$ as the generating series of the characteristics of the representations on the homogeneous components $\mathbb{C}_i[X]$:

$$\mathrm{ch}_t(\mathbb{C}[X]) = \sum_{i=0}^{\infty} \mathrm{ch}(\mathbb{C}_i[X]) t^i. \tag{9}$$

The main result of this section is the following

Theorem 3.3. *Let $(\rho_n)_n$ be the family of local actions of $\mathfrak{S}_n$ related to an equivalence relation $\mathcal{R}_\rho$. Denote by $C := \mathbb{N}/\mathcal{R}_\rho$ the set of equivalence classes. The generating series of the graded characteristic of the actions ρ_n of $\mathfrak{S}_n$ with respect to n is given by*

$$\sum_n \mathrm{ch}_t(\mathbb{C}[X_n]_\rho)u^n = \prod_{c\in C} H\left(\sum_{i\in c} t^i u\right), \tag{10}$$

where,

$$H(u) := \sum_j h_j u^j . \tag{11}$$

In particular, if the number s of equivalence classes in C is finite, then only the products of at most s terms h_i appear in this characteristic. As a consequence, only the irreducible representation V_λ for λ of length smaller than s appear. Thus the action is in fact an action of a quotient of the algebra of $\mathfrak{S}_n$. This can be described explicitly:

Theorem 3.4. *Suppose that the set of equivalence classes associated to a local action ρ of $\mathfrak{S}_n$ is finite of cardinal $s < n$.*

The image of $\mathbb{C}[\mathfrak{S}_n]$ in $\mathrm{End}(\mathbb{C}[X])$ is the quotient of $\mathbb{C}[\mathfrak{S}_n]$ by the ideal generated by

$$\nabla_i := \sum_{\sigma\in\mathfrak{S}^i_{s+1}} -1^{\ell(\sigma)}\sigma, \tag{12}$$

where $\mathfrak{S}^i_{s+1}$ is the shifted symmetric group $\mathfrak{S}_{s+1}$ which acts on the set $\{i, i+1, \dots, i+s+1\}$.

The operators $\rho(\sigma)$ where σ runs along the sets of permutations that avoid the pattern $(s+1)s\dots21$ form a basis of the image of $\mathbb{C}[\mathfrak{S}_n]$ in $\mathrm{End}(\mathbb{C}[X])$.

Recall that the permutation avoiding $s+1s\dots21$ are also the permutations such that the shape of the tableaux obtained by Robinson-Schensted map is a partition with at most s rows.

The proof is then the same as in [7], by comparison on dimension.

3.3. Fixed polynomials and main theorem.

Definition 3. Let ρ be a local action. A polynomial f is *fixed by ρ* or *ρ-symmetric*, if it is invariant under the action of ρ, that is,

$$\rho(\sigma)f = f \qquad \text{for all } \sigma\in\mathfrak{S}_n. \tag{13}$$

The set of ρ-symmetric polynomials is denoted by $\mathbb{K}[X_n]^{\rho(\mathfrak{S}_n)}$.

We now want to know for which local actions the set of fixed polynomials form a sub-algebra of the polynomial algebra $\mathbb{K}[x1, \dots, x_n]$. Equivalently, we want to know under which conditions on $\mathcal{R}$ the product of two fixed polynomials is always a fixed polynomial.

Remark 1. Let ρ be a local action of $\mathfrak{S}_n$ on the space of polynomials. Let $f(x_1, \ldots, x_n)$ be a ρ-symmetric polynomial. Then $f(x_1, \ldots, x_{n-1}, 0)$ is μ-symmetric under the restriction μ of ρ to $\mathfrak{S}_{n-1}$.

We now want to classify the local-actions whose symmetric polynomials form an algebra.

Definition 4. Let r be an integer or infinite. The r-action of $\mathfrak{S}_n$ on polynomials denoted by ρ_r is defined by

$$\rho_r(\sigma_i)(m) = \begin{cases} [\ldots, m_i, m_{i+1}, \ldots] & \text{if } m_i \geq r \text{ and } m_{i+1} \geq r, \\ [\ldots, m_{i+1}, m_i, \ldots] & \text{otherwise.} \end{cases} \tag{14}$$

Note that the trivial action corresponds to $r = 0$, the classical action corresponds to r infinite and the quasi-symmetrizing action of [6, 7] corresponds to $r = 1$.

The main theorem says that they are the only actions whose invariants are stable by multiplication :

Theorem 3.5. *Let ρ be a local action of $\mathfrak{S}_n$. The set $\mathbb{K}[X_n]^{\rho(\mathfrak{S}_n)}$ of fixed polynomials is a sub-algebra of $\mathbb{K}[X]$ if an only if there exists an r, integer or infinite, such that $\rho = \rho_r$.*

Let $\mathrm{QSym}^r(X_n) := \mathbb{K}[X_n]^{\rho_r(\mathfrak{S}_n)}$ denote the sub-algebra of $\mathbb{K}[X_n]$ spanned by r-quasi-symmetric polynomial. It is clear that the restriction morphisms $(n > p)$

$$\begin{aligned} \phi_p^n \ : \ \mathrm{QSym}^r(x_1, \ldots, x_n) &\longrightarrow \mathrm{QSym}^r(x_1, \ldots, x_p) \\ f(x_1, \ldots, x_n) &\longmapsto f(x_1, \ldots, x_p, 0, \ldots, 0) \end{aligned} \tag{15}$$

are compatible (*i.e.* $\phi_n^m \circ \phi_p^n = \phi_p^m$). Hence, it makes sense to take the reverse limit of this projective system in the category of *graded* algebras. We emphasize on the graded condition because we only want to deal with finite degree expressions rather than series. The limit is called the algebra of r-quasi-symmetric functions denoted by QSym^r. We will see in Section 4 than it can actually be equipped naturally with a Hopf algebra structure. For the moment we only concentrate on the algebra structure.

As in the case of classical symmetric and quasi-symmetric functions (*cf.* [10, 4]), QSym^r can be viewed as an algebra of power series of finite degree on an infinite set of variables.

The following inclusions hold:

$$\begin{aligned} \mathbb{K}[X_n] = \mathrm{QSym}^0(X_n) \supset \mathrm{QSym}^1(X_n) \supset \mathrm{QSym}^2(X_n) \supset \\ \cdots \supset \mathrm{QSym}^r(X_n) \supset \cdots \supset \mathrm{QSym}^\infty(X_n) = \mathrm{Sym}(X_n)\,, \end{aligned} \tag{16}$$

and for the infinite alphabet case:

$$\mathrm{QSym}^0 \supset \mathrm{QSym}^1 \supset \cdots \supset \mathrm{QSym}^r \supset \cdots \supset \mathrm{QSym}^\infty = \mathrm{Sym}\ . \tag{17}$$

A basis of QSym^r is given by the analog of the quasi-monomial basis *i.e.* by the orbit-sums. It is indexed by pairs of composition I in parts at least r and a partition λ whose parts are strictly smaller than r. We call such a pair an *r-composition* and write it (I, λ). A monomial whose exponent is an r-composition with possibly added trailing zeroes is called a r-dominant monomial.

For example, here is the list of the 24 3-compositions of 8

$$(8), (7,1), (6,2), (6,11), (53), (5,21), (5,111), (44), (43,1), (4,22),$$
$$(4,211), (4,1111), (35), (34,1), (33,2), (33,11), (3,221), (3,2111),$$
$$(3,11111), (2222), (22211), (221111), (2111111), (11111111)\,.$$

For later reference let us state the following obvious lemma:

Lemma 3.6. *For a fixed r, there is only one r-dominant monomial in each orbit under the r-action.*

The r-dominant monomial corresponding to the orbit of the monomial $m = [m_1, \ldots, m_i, \ldots]$ is the monomial obtained by putting the part m_i strictly smaller than r in decreasing order and the end of m.

Take $I = (I_1, \ldots, I_k)$ and $\lambda = (\lambda_1, \ldots, \lambda_l)$. The monomial function $M^r_{(I,\lambda)}$ is defined as the sum of the orbit of the r-dominant monomial

$$X^{(I,\lambda)} := x_1^{I_1} x_2^{I_2} \ldots x_k^{I_k} x_{k+1}^{\lambda_1} x_{k+2}^{\lambda_2} \ldots x_{k+l}^{\lambda_l} \tag{18}$$

under the r-action, that is

$$M^r_{(I,\lambda)}(X) := \sum_{\substack{i_1 \neq i_2 \neq \cdots \neq i_{k+l} \\ i_1 < i_2 < \cdots < i_k}} x_{i_1}^{I_1} x_{i_2}^{I_2} \ldots x_{i_k}^{I_k} x_{i_{k+1}}^{\lambda_1} x_{i_{k+2}}^{\lambda_2} \ldots x_{i_{k+l}}^{\lambda_l} \tag{19}$$

One easily sees that

$$M^r_{(I,\lambda)} = \sum_{K \in I \sqcup\!\sqcup \lambda^\sigma} M_K\,, \tag{20}$$

where λ^σ denotes the set of all distinct reordering of λ and $M_K = M^1_K$ is the classical quasi-symmetric functions. For example

$$\begin{aligned} M^3_{(4,6),(2,1)} &= M_{4,6,1,2} + M_{4,1,6,2} + M_{4,1,2,6} + M_{1,4,6,2} + M_{1,4,2,6} + M_{1,2,4,6} \\ &\quad + M_{4,6,2,1} + M_{4,2,6,1} + M_{4,2,1,6} + M_{2,4,6,1} + M_{2,4,1,6} + M_{2,1,4,6} \end{aligned}$$

The previous expression allows to deduce the product rule of QSym^r from the one of QSym. Recall that the product of Classical quasi-monomial function is given by the quasi-shuffle:

$$M_I M_J = \sum_K \langle K | I \uplus J \rangle\, M_K\,, \tag{21}$$

where $I \uplus J$ is defined recursively by

$$\begin{aligned} \epsilon \uplus I &= I \uplus \epsilon = I && (22) \\ I \uplus J &= i_1 \triangleright (I' \uplus J) + j_1 \triangleright (I \uplus J') + (i_1 + j_1) \triangleright (I' \uplus J')\,, && (23) \end{aligned}$$

where ϵ is the empty composition, $\triangleright$ denotes the operation of adding a part at the beginning of a composition, and finally $I = i_1 \triangleright I'$ and $J = j_1 \triangleright J'$. Then

Proposition 3.7. *Let $r > 0$ and (I, λ) and J, μ two r-compositions. Then,*

$$M^r_{(I,\lambda)} M^r_{(J,\mu)} = \sum_{(K,\nu)} \sum_{\substack{A \in I \sqcup\!\sqcup \lambda^\sigma \\ B \in J \sqcup\!\sqcup \mu^\sigma}} \langle (K,\nu) | A \uplus B \rangle \, M^r_{(K,\nu)} \tag{24}$$

where λ^σ and μ^σ run through the set of the reordering of λ and μ as in Equation (20) and $\langle (K,\nu) | A \uplus B \rangle$ is the coefficient of (K,ν) in the quasi-shuffle of A with B.

3.4. Generating series.

Theorem 3.8. *The generating series of the graded characteristic of the r-action on $\mathfrak{S}_n$ with respect to n is given by*

$$\sum_n \mathrm{ch}_t(\mathbb{C}[X_n]) a^n = H\left(\frac{at^r}{1-t}\right) \prod_{i=0}^{r-1} H(at^i)\,, \tag{25}$$

where as usual

$$H(u) := \sum_i h_i u^i = \sigma_u(A) = \sum_i h_i(A) u^i\,. \tag{26}$$

A consequence of this theorem, which can also be proved directly, is the generating series of the homogeneous dimension of QSym^r.

Theorem 3.9. *The generating series of the dimensions of the homogeneous components of $\mathrm{QSym}^r(X_n)$ is given by:*

$$\sum_{d,n} \dim_d(\mathrm{QSym}^r(X_n)) a^n t^d = \frac{1-t}{1-t-at^r} \prod_{i=0}^{r-1} \frac{1}{1-at^i} \tag{27}$$

The generating series of the dimensions of the homogeneous components of QSym^r is given by:

$$\mathrm{Hilb}_t(\mathrm{QSym}^r) := \sum_{d,n} \dim_d(\mathrm{QSym}^r) t^d = \frac{1-t}{1-t-t^r} \prod_{i=1}^{r-1} \frac{1}{1-t^i} \tag{28}$$

Here are the first values:

	0	1	1	3	4	5	6	7	8	9	10
QSym	1	1	2	4	8	16	32	64	128	256	512
QSym^2	1	1	2	3	5	8	13	21	34	55	89
QSym^3	1	1	2	3	5	7	11	16	24	35	52
QSym^4	1	1	2	3	5	7	11	15	22	31	44
Sym	1	1	2	3	5	7	11	15	22	30	42

4. Hopf algebra structure

There are two equivalent ways to endow QSym^r with a Hopf algebra structure. The first way is to prove that QSym^r is a sub-Hopf algebra of QSym; that is that the coproduct of an r-Quasi-Symmetric element is an r-quasi-symmetric element. We prefer a slightly less direct but in our opinion more instructive way, namely to use the classical alphabet doubling trick to define the coproduct.

Let $X = \{x_1, x_2, \dots\}$ and $Y = \{y_1, y_2, \dots\}$ be two totally ordered infinite alphabets. The ordered sum $X \sqcup Y$ of alphabet is, as a set, the union of X and Y, where we keep the order inside X and Y and we choose that the elements of X are smaller than the elements of Y. Clearly, as soon as an alphabet Z is totally ordered and infinite $\mathrm{QSym}^r \simeq \mathrm{QSym}^r(Z)$; in particular,

$$\mathrm{QSym}^r(X) \simeq \mathrm{QSym}^r(X \sqcup Y)\,. \tag{29}$$

But now its clear from the restriction property (Remark 1) that a r-quasi-symmetric function in $X \sqcup Y$ is in particular r-quasi-symmetric in X and in Y. Thus one has a natural algebra morphism

$$\mathrm{QSym}^r(X) \mapsto \mathrm{QSym}^r(X) \otimes \mathrm{QSym}^r(Y)\,. \tag{30}$$

which can be seen as a coproduct. The co-associativity is an easy consequence of the associativity of the ordered sum of alphabets. The co-unit corresponds to the restriction to an empty alphabet, *i.e.* the mapping which extracts the coefficient of the constant monomial.

Hence the following theorem

Theorem 4.1. *The r-quasi-symmetric functions form an infinite hierarchy of graded sub-Hopf-algebras:*

$$\mathrm{QSym} = \mathrm{QSym}^1 \supset \cdots \supset \mathrm{QSym}^r \supset \cdots \supset \mathrm{QSym}^\infty = \mathrm{Sym}\,. \tag{31}$$

Let us express this explicitly.

Proposition 4.2. *Let $r > 0$ and (I, λ) be a r-composition. Then, the coproduct of $M^r_{(I,\lambda)}$ is given by*

$$\delta(M^r_{(I,\lambda)}) = \sum_{(K,K')=I;\ \lambda\in\nu\sqcup\!\sqcup\nu'} M^r_{(K,\nu)} \otimes M^r_{(K',\nu')}\,. \tag{32}$$

For example

$$\begin{aligned}\delta(M^2_{635211}) = {} & M^2_{635211}\otimes 1 + M^2_{63521}\otimes M^2_1 + M^2_{63511}\otimes M^2_2 + M^2_{6351}\otimes M^2_{21} + \\ & M^2_{635}\otimes M^2_{211} + M^2_{63211}\otimes M^2_5 + M^2_{6321}\otimes M^2_{51} + M^2_{6311}\otimes M^2_{52} + \\ & M^2_{631}\otimes M^2_{521} + M^2_{63}\otimes M^2_{5211} + M^2_{6211}\otimes M^2_{35} + M^2_{621}\otimes M^2_{351} + \\ & M^2_{611}\otimes M^2_{352} + M^2_{61}\otimes M^2_{3521} + M^2_{6}\otimes M^2_{35211} + M^2_{211}\otimes M^2_{635} + \\ & M^2_{21}\otimes M^2_{6351} + M^2_{11}\otimes M^2_{6352} + M^2_{1}\otimes M^2_{63521} + 1\otimes M^2_{635211}\end{aligned}$$

4.1. **Dual Hopf algebra and noncommutative symmetric functions.** Since the Hopf algebra QSym^r is a sub-algebra the algebra of quasi-symmetric functions, its dual is naturally a quotient of **NCSF**. The goal of this section is to describe explicitly this duality.

Recall that **NCSF** is the free associative algebra over an infinite set of generators S_i with basis (S_i) where

$$S^I = S^{(I_1,I_2,\dots,I_k)} := S_{I_1}S_{I_2}\dots S_{I_k}\,. \tag{33}$$

The bases (M_I) and (S^I) are two dual bases of respectively QSym and **NCSF**. Since the basis $M^r_{(I,\lambda)}$ of QSym^r is formed by disjoints sums of elements of (M_I), the dual of QSym^r can be described by identifying some elements of the basis (S^I).

The dual of QSym^r can be described explicitly as follows:

Theorem 4.3. *The dual of* QSym^r *is the quotient of the Hopf algebra* **NCSF** *by the relations*

$$S_iS_j = S_jS_i \qquad \text{for all } i < r \text{ and } j \in \mathbb{N}. \tag{34}$$

A basis of this quotient is given by $(S^{(I,\lambda)})$, where (I,λ) goes along the set of all r-compositions.

4.2. **Primitive Elements.** The Hopf algebra QSym^r is graded connected and co-commutative. Hence, by the Milnor-Moore theorem it is isomorphic to the universal enveloping algebra of the Lie algebra of its primitive elements. The goal of this section is to explicitly describe these primitive elements.

In [3] (see also [11]), an explicit construction of several generating families, called non-commutative power sums of the Lie algebra Prim(**NCSF**) is given. Let us recall briefly one of them. We use a slightly modified base: our Φ_n corresponds to $n\Phi_n$ of [3] or equivalently to the base P^*_n of [11]. They are defined by means of generating series, the relation

$$\sum_{k\geq 1} t^k\Phi_k = \log\left(1+\sum_{k\geq 1} S_k t^k\right) \tag{35}$$

defining uniquely a sequence of primitive elements $(\Phi_k)_{k\geq 1}$ in **NCSF**. Explicitly this gives:

$$\Phi_n = \sum_{K\vDash n} \frac{(-1)^{\ell(K)-1}}{\ell(k)} S^K\,. \tag{36}$$

An important consequence is that **NCSF** is freely generated by (Φ_n); moreover as a Hopf algebra it is isomorphic to the universal enveloping algebra $\mathrm{U}(\mathcal{L}(\Phi_i; i>0))$ where $\mathcal{L}(\Phi_i; i>0)$ is the free Lie algebra generated by $(\Phi_i)_{i>0}$.

By projecting these elements into $\mathbf{NCSF}^r$ we get an explicit description of the structure of the primitive Lie algebra $\mathrm{Prim}(\mathbf{NCSF}^r)$ and of the Hopf algebra $\mathbf{NCSF}^r$.

Theorem 4.4. $\mathbf{NCSF}^r$ *is the quotient of the Hopf algebra* $\mathbf{NCSF}$ *by the relations*

$$[\Phi_i, \Phi_j] = 0 \qquad \text{for all } i < r \text{ and } j \in \mathbb{N}. \tag{37}$$

The Lie algebra $\mathrm{Prim}(\mathbf{NCSF}^r)$ *is generated by* $(\Phi_i)_{i>0}$. *Moreover it is isomorphic to the central extension of the free Lie algebra with generators* $(\Phi_i)_{i\geq r}$ *by the trivial commutative Lie algebra with basis* $(\Phi_i)_{i<r}$

Hence we get that the set of the elements

$$\Phi^{(I,\lambda)} = \Phi_{I_1} \dots \Phi_{I_k} \Phi_{\lambda_1} \dots \Phi_{\lambda_k}\,, \tag{38}$$

where (I,λ) goes along all r-compositions, is a multiplicative basis of $\mathbf{NCSF}^r$ generated by primitive elements.

5. Free algebra structure

This last property shows that the dual basis $\phi_{(I,\lambda)}$ has a very nice product rule. As a consequence, we elucidate the structure of QSym^r. First let us give explicitly the dual basis of Φ^I of $\mathbf{NCSF}^r$. In the classical case of the duality $(\mathrm{QSym}, \mathbf{NCSF})$ the dual bases of the Φ^I is the basis ϕ_I given by

$$\phi_I = \sum_{I \succeq K} \frac{1}{(\#(I,K))!} M_K\,, \tag{39}$$

where $(\#(I,K))! = \ell(J_1)!\,\ell(J_2)! \dots \ell(J_r)!$, and $J_1 \dots J_r$ are compositions such that $I = I_1 \cdot I_2 \cdots I_r$ and $K = (|I_1|, |I_2|, \dots, |I_r|)$. In particular ϕ_i is equal to the symmetric power sum $p_i = \sum x_k^i$. Since the Φ_i's are primitive the product of the ϕ_I is given by the shuffle product

$$\phi_I \phi_J = \sum_K \langle K | I \sqcup\!\sqcup J \rangle\, \phi_K\,. \tag{40}$$

Let $\phi'^r_{(I,\lambda)}$ be the dual basis of $\Phi^{(I,\lambda)}$. The formula which expresses $\phi_{(I,\lambda)}$ in terms of the ϕ_J is the same as the formula which expresses $M_{(I,\lambda)}$ in terms of the M_J's $\phi'^r_{(I,\lambda)} = \sum_{K \in I \sqcup\!\sqcup \lambda^\sigma} \phi_K$. We prefer a slightly different normalization: write λ in exponential notation $\lambda = (1^{m_1} 2^{m_1} \dots)$, and set $z_\lambda := \prod m_i!$. The basis $\phi^r_{(I,\lambda)}$ is defined as

$$\phi^r_{(I,\lambda)} := z_\lambda \phi'^r_{(I,\lambda)} = z_\lambda \sum_{K \in I \sqcup\!\sqcup \lambda^\sigma} \phi_K = \phi_I \phi_{\lambda_1} \phi_{\lambda_2} \dots \phi_{\lambda_l}\,. \tag{41}$$

Then the product of two such functions is given by

$$\phi^r_{(I,\lambda)} \phi^r_{(J,\mu)} = \sum_K \langle K | I \sqcup\!\sqcup J \rangle\, \phi^r_{(K,\nu)}\,, \tag{42}$$

where ν is the partition obtained by sorting the concatenation of the partitions λ and μ. Now it is known that the shuffle algebra is free on the Lyndon words (see [12] for a definition of Lyndon words). Hence we have the following theorem

Theorem 5.1. *The algebra* QSym^r *is the free algebra generated by the set*

- ϕ_k *for* $k < r$*;*
- ϕ_L *for all Lyndon words* $L = (l_1 \dots l_m)$ *with* $l_i \geq r$*.*

As a simple consequence:

Corollary 5.2. QSym^r *is a free* Sym*-module with generators* $(\phi_L)_L$ *where* L *goes along the set of Lyndon words* $L = (l_1 \dots l_m)$ *with* $m > 1$ *and* $l_i \geq r$*.*

It seems that the same property holds when the number of variables is finite. This is our main conjecture.

Conjecture 1. Let n be finite and $X_n := \{x_1, \dots, x_n\}$. As a $\mathrm{Sym}(X_n)$-module $\mathrm{QSym}^r(X_n)$ is free of dimension $n!$.

As proved recently by Garsia and Wallach [2], this conjecture holds for in the case of QSym that is for $r = 1$. It seems that their proof can be adapted to the generalized case, but we have not worked out this proof. However, we believe that the knowledge of the generalized case and the following proposition allows us to simplify the proof. Assuming this conjecture, we get the following Hilbert series for the quotient:

Proposition 5.3. *The quotient of the Hilbert series of* QSym^r *and* Sym *is given by*

$$\mathrm{Hilb}_t(\mathrm{QSym}^r(X_n)) / \mathrm{Hilb}_t(\mathrm{Sym}(X_n)) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{Maj}(\sigma) + r(n - \mathrm{Fix}(\sigma))}, \tag{43}$$

where $\mathrm{Fix}(\sigma)$ *is the number of fixed points of the permutation* σ *and* $\mathrm{Maj}(\sigma)$ *is the* Major index *of* σ *that is the sum if the descents of* σ*.*

As far as we now, such a formula was unknown even for QSym. It follows from a result of Gessel and Reutenauer [5] (Theorem (8.4)).

This identity strongly suggests that the freeness of as well as the $n!$ result for each $Qsym^r[X_n]$ should be derivable by means of some action of S_n, in the same manner these results are established for the the polynomial ring. To this date no such a proof has been found even in the $r = 1$ case since the Garsia-Wallach proof follows a completely different path. Identifying the action that yields such a proof would be an outstanding result that would deeply increase our understanding of these remarkable modules.

References

[1] Duchamp G., Krob D., Leclerc B. ,Thibon J.-Y., *Fonctions quasi-symétriques, fonctions symétriques non commutatives, et algèbres de Hecke à $q = 0$*, C.R. Acad. Sci. Paris, **322**, 107–112, 1996.

[2] A. Garsia et N. Wallach, *Qsym over Sym is free*, J. Combin. Theory, Ser. A, **104** (2003), no. 2, 217–263.

[3] Gelfand I.M., Krob D., Leclerc B., Lascoux A., Retakh V.S., Thibon J.-Y., *Noncommutative symmetric functions*, Adv. in Math., **112**, 218–348, 1995.

[4] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, *in* “Combinatorics and algebra”, *C. Greene, ed.*, *Contemporary Mathematics* **34** (1984), 289–301.

[5] I. Gessel, C. Reutenauer, Counting permutations with cycle and descent structure, J. Combin. Theory, Ser A, **64** (1993), 189–215.

[6] F. Hivert, *Combinatoire des fonctions quasi-symétriques*, Phd Thesis, University of Marne-La-Vallée, (1999).

[7] F. Hivert, *Hecke Algebras, Differences Operators, and Quasi-Symmetric Functions*, Adv. Math. **(155)**, (2000), 181-238.

[8] Krob D., Thibon J.-Y., *Noncommutative symmetric functions IV : Quantum linear groups and Hecke algebras at $q = 0$*, J. Alg. Comb., **6** (1997), no. 4, 339–376.

[9] A. Lascoux, and M. P. Schützenberger, Le Monode plaxique, *Quad. del. Ric. Sci.* **109** (1981), 129–156.

[10] I. G. Macdonald, Symmetric functions and Hall polynomials, *Clarendon Press*, (1995).

[11] Malvenuto C., Reutenauer C., *Duality between quasi-symmetric functions and Solomon descent algebra*, J. Algebra, **177**, 967–982, 1995.

[12] C. Reutenauer, *Free Lie Algebras*, Oxford science publications, (1993).