

# QUASI-INVARIANT AND SUPER-COINVARIANT POLYNOMIALS FOR THE GENERALIZED SYMMETRIC GROUP

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ABSTRACT. The aim of this work is to extend the study of super-coinvariant polynomials, introduced in [2, 3], to the case of the generalized symmetric group  $G_{n,m}$ , defined as the wreath product  $C_m \wr \mathcal{S}_n$  of the symmetric group by the cyclic group. We define a quasi-symmetrizing action of  $G_{n,m}$  on  $\mathbb{Q}[x_1, \dots, x_n]$ , analogous to those defined in [12] in the case of  $\mathcal{S}_n$ . The polynomials invariant under this action are called quasi-invariant, and we define super-coinvariant polynomials as polynomials orthogonal, with respect to a given scalar product, to the quasi-invariant polynomials with no constant term. Our main result is the description of a Gröbner basis for the ideal generated by quasi-invariant polynomials, from which we deduce that the dimension of the space of super-coinvariant polynomials is equal to  $m^n C_n$  where  $C_n$  is the  $n$ -th Catalan number.

RÉSUMÉ. Le but de ce travail est d'étendre l'étude des polynômes super-coinvariants (définis dans [2]), au cas du groupe symétrique généralisé  $G_{n,m}$ , défini comme le produit en couronne  $C_m \wr \mathcal{S}_n$  du groupe symétrique par le groupe cyclique. Nous définissons ici une action quasi-symétrisante de  $G_{n,m}$  sur  $\mathbb{Q}[x_1, \dots, x_n]$ , analogue à celle définie dans [12] dans le cas de  $\mathcal{S}_n$ . Les polynômes invariants sous cette action sont dits quasi-invariants, et les polynômes super-coinvariants sont les polynômes orthogonaux aux polynômes quasi-invariants sans terme constant (pour un certain produit scalaire). Notre résultat principal est l'obtention d'une base de Gröbner pour l'idéal engendré par les polynômes quasi-invariants. Nous en déduisons alors que la dimension de l'espace des polynômes super-coinvariants est  $m^n C_n$  où  $C_n$  est le  $n$ -ième nombre de Catalan.

## 1. INTRODUCTION

Let  $X$  denote the alphabet in  $n$  variables  $(x_1, \dots, x_n)$  and  $\mathbb{C}[X]$  denote the space of polynomials with complex coefficients in the alphabet  $X$ . Let  $G_{n,m} = C_m \wr \mathcal{S}_n$  denote the wreath product of the symmetric group  $\mathcal{S}_n$  by the cyclic group  $C_m$ . This group is sometimes known as the *generalized symmetric group* (cf. [17]). It may be seen as the group of  $n \times n$  matrices in which each row and each column has exactly one non-zero entry (pseudo-permutation matrices), and such that the non-zero entries are  $m$ -th roots of unity. The order of  $G_{n,m}$  is  $m^n n!$ . When  $m = 1$ ,  $G_{n,m}$  reduces to the symmetric group  $\mathcal{S}_n$ , and when  $m = 2$ ,  $G_{n,m}$  is the hyperoctahedral group  $B_n$ , *i.e.* the group of signed permutations, which is the Weyl group of type  $B$  (see [14])

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for example for further details). The group  $G_{n,m}$  acts classically on  $\mathbb{C}[X]$  by the rule

$$(1.1) \quad \forall g \in G_{n,m}, \forall P \in \mathbb{C}[X], g.P(X) = P(X.^t g),$$

where  $g$  is the transpose of the matrix  $g$  and  $X$  is considered as a row vector. Let

$$Inv_{n,m} = \{P \in \mathbb{C}[X] / \forall g \in G_{n,m}, g.P = P\}$$

denote the set of  $G_{n,m}$ -invariant polynomials. Let us denote by  $Inv_{n,m}^+$  the set of such polynomials with no constant term. We consider the following scalar product on  $\mathbb{C}[X]$ :

$$(1.2) \quad \langle P, Q \rangle = P(\partial X)Q(X) |_{X=0}$$

where  $\partial X$  stands for  $(\partial x_1, \dots, \partial x_n)$  and  $X = 0$  stands for  $x_1 = \dots = x_n = 0$ . The space of  $G_{n,m}$ -coinvariant polynomials is then defined by

$$\begin{aligned} Cov_{n,m} &= \{P \in \mathbb{C}[X] / \forall Q \in Inv_{n,m}, Q(\partial X)P = 0\} \\ &= \langle Inv_{n,m}^+ \rangle^\perp \simeq \mathbb{C}[X] / \langle Inv_{n,m}^+ \rangle \end{aligned}$$

where  $\langle S \rangle$  denotes the ideal generated by a subset  $S$  of  $\mathbb{C}[X]$ .

A classical result of Chevalley [6] states the following equality:

$$(1.3) \quad \dim Cov_{n,m} = |G_{n,m}| = m^n n!$$

which reduces when  $m = 1$  to the theorem of Artin [1] that the dimension of the harmonic space  $\mathbf{H}_n = Cov_{n,1}$  (cf. [9]) is  $n!$ .

Our aim is to give an analogous result in the case of quasi-symmetrizing action. The ring  $Qsym$  of quasi-symmetric functions was introduced by Gessel [11] as a source of generating functions for  $P$ -partitions [18] and appears in more and more combinatorial contexts [5, 18, 19]. Malvenuto and Reutenauer [16] proved a graded Hopf duality between  $QSym$  and the Solomon descent algebras and Gelfand *et. al.* [10] defined the graded Hopf algebra  $NC$  of non-commutative symmetric functions and identified it with the Solomon descent algebra.

In [2, 3], Aval *et. al.* investigated the space  $\mathbf{SH}_n$  of super-coinvariant polynomials for the symmetric group, defined as the orthogonal (with respect to (1.2)) of the ideal generated by quasi-symmetric polynomials with no constant term, and proved that its dimension as a vector space equals the  $n$ -th Catalan number:

$$(1.4) \quad \dim \mathbf{SH}_n = C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Our main result is a generalization of the previous equation in the case of super-coinvariant polynomials for the group  $G_{n,m}$ .

In Section 2, we define and study a “quasi-symmetrizing” action of  $G_{n,m}$  on  $\mathbb{C}[X]$ . We also introduce invariant polynomials under this action, which are called quasi-invariant, and polynomials orthogonal to quasi-invariant polynomials, which are called super-coinvariant. The Section 3 is devoted to the proof of our main result (Theorem 2.4), which gives the dimension of the space  $SCov_{n,m}$  of super-coinvariant polynomials for  $G_{n,m}$ : we construct an explicit basis for  $SCov_{n,m}$  from which we deduce its Hilbert series.

2. A QUASI-SYMMETRIZING ACTION OF  $G_{n,m}$

We use vector notation for monomials. More precisely, for  $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ , we denote  $X^\nu$  the monomial

$$(2.1) \quad x_1^{\nu_1} x_2^{\nu_2} \cdots x_n^{\nu_n}.$$

For a polynomial  $P \in \mathbb{Q}[X]$ , we further denote  $[X^\nu]P(X)$  as the coefficient of the monomial  $X^\nu$  in  $P(X)$ .

Our first task is to define a quasi-symmetrizing action of the group  $G_{n,m}$  on  $\mathbb{C}[X]$ , which reduces to the quasi-symmetrizing action of Hivert (*cf.* [12]) in the case  $n = 1$ . This is done as follows. Let  $A \subset X$  be a subalphabet of  $X$  with  $l$  variables and  $K = (k_1, \dots, k_l)$  be a vector of positive ( $> 0$ ) integers. If  $B$  is a vector whose entries are distinct variables  $x_i$  multiplied by roots of unity, the vector  $(B)_<$  is obtained by ordering the elements in  $B$  with respect to the variable order. Now the quasi-symmetrizing action of  $g \in G_{n,m}$  is given by

$$(2.2) \quad g \bullet A^K = w(g)^{c(K)} (A.^t |g|)_<^K$$

where  $w(g)$  is the weight of  $g$ , *i.e.* the product of its non-zero entries,  $|g|$  is the matrix obtained by taking the modules of the entries of  $g$ , and the coefficient  $c(K)$  is defined as follows:

$$c(K) = \begin{cases} 0 & \text{if } \forall i, k_i \equiv 0 [m] \\ 1 & \text{if not.} \end{cases}$$

**Example 2.1.** If  $m = 3$  and  $n = 3$ , and we denote by  $j$  the complex number  $j = e^{\frac{2i\pi}{3}}$ , then for example

$$\begin{aligned} & \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix} \bullet (x_1^2 x_2) \\ &= (j^2)^1 \left[ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot (x_1, x_2) \right]_{<}^{(2,1)} \\ &= j^2 (x_3, x_1)_{<}^{(2,1)} \\ &= j^2 (x_1, x_3)^{(2,1)} \\ &= j^2 x_1^2 x_3. \end{aligned}$$

It is clear that this defines an action of the generalized symmetric group  $G_{n,m}$  on  $\mathbb{C}[X]$ , which reduces to Hivert's quasi-symmetrizing action (*cf.* [12], Proposition 3.4) in the case  $m = 1$ .

Let us now study its invariant and coinvariant polynomials. We need to recall some definitions.

A *composition*  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$  of a positive integer  $d$  is an ordered list of positive integers ( $> 0$ ) whose sum is  $d$ . For a *vector*  $\nu \in \mathbb{N}^n$ , let  $c(\nu)$  represent the composition obtained by erasing zeros (if any) in  $\nu$ . A polynomial  $P \in \mathbb{Q}[X]$  is said

to be *quasi-symmetric* if and only if, for any  $\nu$  and  $\mu$  in  $\mathbb{N}^n$ , we have

$$[X^\nu]P(X) = [X^\mu]P(X)$$

whenever  $c(\nu) = c(\mu)$ . The space of quasi-symmetric polynomials in  $n$  variables is denoted by  $QSym_n$ .

The polynomials invariant under the action (2.2) of  $G_{n,m}$  are said to be *quasi-invariant* and the space of quasi-invariant polynomials is denoted by  $QInv_{n,m}$ , i.e.

$$P \in QInv_{n,m} \Leftrightarrow \forall g \in G_{n,m}, g \bullet P = P.$$

Let us recall (cf. [12], Proposition 3.15) that  $QInv_{n,1} = QSym_n$ . The following proposition gives a characterization of  $QInv_{n,m}$ .

**Proposition 2.2.** *One has*

$$P \in QInv_{n,m} \Leftrightarrow \exists Q \in QSym_n / P(X) = Q(X^m)$$

where  $Q(X^m) = Q(x_1^m, \dots, x_n^m)$ .

*Proof.* Let  $P$  be an element of  $QInv_{n,m}$ . Let us denote by  $\zeta$  the  $m$ -th root of unity  $\zeta = e^{\frac{2i\pi}{m}}$  and by  $g_j$  the element of  $G_{n,m}$  whose matrix is

$$\begin{pmatrix} \zeta & & 0 & \\ & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$$

with the  $\zeta$  in place  $j$ . Then we observe that the identities

$$\forall j = 1, \dots, n, \frac{1}{m}(P + g_j \bullet P + g_j^2 \bullet P + \dots + g_j^{m-1} \bullet P) = P$$

imply that every exponents appearing in  $P$  are multiples of  $m$ . Thus there exists a polynomial  $Q \in \mathbb{C}[X]$  such that  $P(X) = Q(X^m)$ . To conclude, we note that  $\mathcal{S}_n \subset G_{n,m}$  implies that  $P$  is quasi-symmetric, whence  $Q$  is also quasi-symmetric.

The reverse implication is obvious.  $\square$

Let us now define *super-coinvariant* polynomials:

$$\begin{aligned} SCov_{n,m} &= \{P \in \mathbb{C}[X] / \forall Q \in QInv_{n,m}, Q(\partial X)P = 0\} \\ &= \langle QInv_{n,m}^+ \rangle^\perp \simeq \mathbb{C}[X] / \langle QInv_{n,m}^+ \rangle \end{aligned}$$

with the scalar product defined in (1.2). This is the natural analogous to  $Cov_n$  in the case of quasi-symmetrizing actions and  $SCov_{n,m}$  reduces to the space of super-harmonic polynomials  $\mathbf{SH}_n$  (cf. [3]) when  $m = 1$ .

**Remark 2.3.** It is clear that any polynomial invariant under (2.2) is also invariant under (1.1), i.e.  $Inv_{n,m} \subset QInv_{n,m}$ . By taking the orthogonal, this implies that  $SCov_{n,m} \subset Cov_{n,m}$ . These observations somewhat justify the terminology.

Our main result is the following theorem which is a generalization of equality (1.4).

**Theorem 2.4.** *The dimension of the space  $Scov_{n,m}$  is given by*

$$(2.3) \quad \dim SCov_{n,m} = m^n C_n = m^n \frac{1}{n+1} \binom{2n}{n}.$$

**Remark 2.5.** In the case of the hyperoctahedral group  $B_n = G_{n,2}$ , C.-O. Chow [7] defined a class  $BQSym(x_0, X)$  of quasi-symmetric functions of type  $B$  in the alphabet  $(x_0, X)$ . His approach is quite different from ours. In particular, one has the equality:

$$BQSYm(x_0, X) = QSym(X) + QSym(x_0, X).$$

In the study of the coinvariant polynomials, it is not difficult to prove that the quotient  $\mathbb{C}[x_0, X]/\langle BQSym^+ \rangle$  is isomorphic to the quotient  $\mathbb{C}[X]/\langle QSym^+ \rangle$  studied in [3]. To see this, we observe that if  $\mathcal{G}$  is the Gröbner basis of  $\langle QSym^+ \rangle$  constructed in [3] (see also the next section), then the set  $\{x_0, \mathcal{G}\}$  is a Gröbner basis (any syzygy is reducible thanks to Buchberger’s first criterion, cf. [8]).

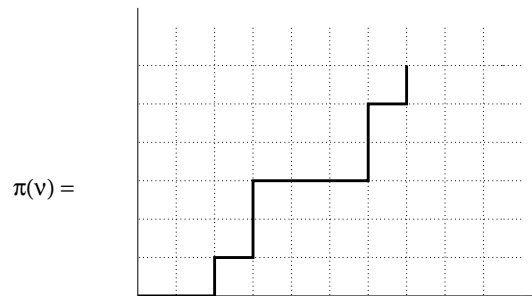
The next section is devoted to give a proof of Theorem 2.4 by constructing an explicit basis for the quotient  $\mathbb{C}[X]/\langle QInv_{n,m}^+ \rangle$ .

### 3. PROOF OF THE MAIN THEOREM

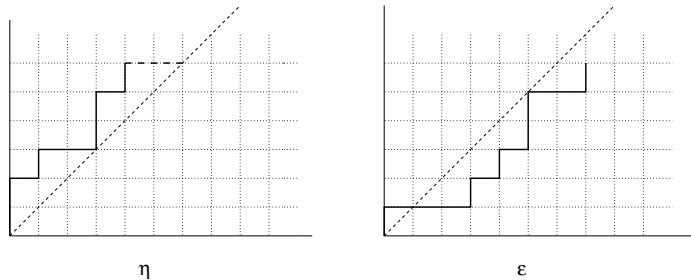
Our task is here to construct an explicit monomial basis for the quotient space  $\mathbb{C}[X]/\langle QInv_{n,m}^+ \rangle$ . Let us first recall (cf. [3]) the following bijection which associates to any vector  $\nu \in \mathbb{N}^n$  a path  $\pi(\nu)$  in the  $\mathbb{N} \times \mathbb{N}$  plane with steps going north or east as follows. If  $\nu = (\nu_1, \dots, \nu_n)$ , the path  $\pi(\nu)$  is

$$(0, 0) \rightarrow (\nu_1, 0) \rightarrow (\nu_1, 1) \rightarrow (\nu_1 + \nu_2, 1) \rightarrow (\nu_1 + \nu_2, 2) \rightarrow \dots \\ \rightarrow (\nu_1 + \dots + \nu_n, n - 1) \rightarrow (\nu_1 + \dots + \nu_n, n).$$

For example the path associated to  $\nu = (2, 1, 0, 3, 0, 1)$  is



We distinguish two kinds of paths, thus two kinds of vectors, with respect to their “behavior” regarding the diagonal  $y = x$ . If the path remains above the diagonal, we call it a *Dyck path*, and say that the corresponding vector is *Dyck*. If not, we say that the path (or equivalently the associated vector) is *transdiagonal*. For example  $\eta = (0, 0, 1, 2, 0, 1)$  is Dyck and  $\varepsilon = (0, 3, 1, 1, 0, 2)$  is transdiagonal.



We then have the following result which generalizes Theorem 4.1 of [3] and which clearly implies the Theorem 2.4.

**Theorem 3.1.** *The set of monomials*

$$\mathcal{B}_{n,m} = \{(X_n)^{m\eta+\alpha} / \pi(\eta) \mid \pi(\eta) \text{ is a Dyck path}, 0 \leq \alpha_i < m\}$$

is a basis for the quotient  $\mathbb{C}[X_n] / \langle \text{QInv}_{n,m}^+ \rangle$ .

To prove this result, the goal is here to construct a Gröbner basis for the ideal  $\mathcal{J}_{n,m} = \langle \text{QInv}_{n,m}^+ \rangle$ . We shall use results of [2, 3].

Recall that the *lexicographic order* on monomials is

$$(3.1) \quad X^\nu >_{\text{lex}} X^\mu \quad \text{iff} \quad \nu >_{\text{lex}} \mu,$$

if and only if the first non-zero part of the vector  $\nu - \mu$  is positive.

For any subset  $\mathcal{S}$  of  $\mathbb{Q}[X]$  and for any positive integer  $m$ , let us introduce  $\mathcal{S}^m = \{P(X^m), P \in \mathcal{S}\}$ . If we denote by  $G(I)$  the unique reduced monic Gröbner basis (cf. [8]) of an ideal  $I$ , then the simple but crucial fact in our context is the following.

**Proposition 3.2.** *With the previous notations,*

$$(3.2) \quad G(\langle \mathcal{S}^m \rangle) = G(\langle \mathcal{S} \rangle)^m.$$

*Proof.* This is a direct consequence of Buchberger's criterion. Indeed, if for every pair  $g, g'$  in  $G(\langle \mathcal{S} \rangle)$ , the syzygy

$$S(g, g')$$

reduces to zero, then the syzygy

$$S(g(X^m), g'(X^m))$$

also reduces to zero in  $G(\langle \mathcal{S}^m \rangle)$  by exactly the same computation.  $\square$

Let us recall that in [2] is constructed a family  $\mathcal{G}$  of polynomials  $G_\varepsilon$  indexed by transdiagonal vectors  $\varepsilon$ . This family is constructed by using recursive relations of the fundamental quasi-symmetric functions and one of its property (cf. [2]) says that the leading monomial of  $G_\varepsilon$  is:  $LM(G_\varepsilon) = X^\varepsilon$ . Since  $\mathcal{G}$  is a Gröbner basis of  $\mathcal{J}_{n,1}$ , the following result is a consequence of Propositions 2.2 and 3.2.

**Proposition 3.3.** *The set  $\mathcal{G}^m$  is a Gröbner basis of the ideal  $\mathcal{J}_{n,m}$ .*

To conclude the proof of Theorem 3.1, it is sufficient to observe that the monomials not divisible by a leading monomial of an element of  $\mathcal{G}^m$ , i.e. by a  $X^{m\varepsilon}$  for  $\varepsilon$  transdiagonal, are precisely the monomials appearing in the set  $\mathcal{B}_{n,m}$ .

As a corollary of Theorem 3.1, one gets an explicit formula for the Hilbert series of  $SCov_{n,m}$ . For  $k \in \mathbb{N}$ , let  $SCov_{n,m}^{(k)}$  denote the projection

$$(3.3) \quad SCov_{n,m}^{(k)} = SCov_{n,m} \cap \mathbb{Q}^{(k)}[X]$$

where  $\mathbb{Q}^{(k)}[X]$  is the vector space of homogeneous polynomials of degree  $k$  together with zero.

Let us denote by  $F_{n,m}(t)$  the Hilbert series of  $SCov_{n,m}$ , *i.e.*

$$(3.4) \quad F_{n,m}(t) = \sum_{k \geq 0} \dim SCov_{n,m}^{(k)} t^k.$$

Let us recall that in [3] is given an explicit formula for  $F_{n,1}$ :

$$(3.5) \quad F_{n,1}(t) = F_n(t) = \sum_{k=0}^{n-1} \frac{n-k}{n+k} \binom{n+k}{k} t^k$$

using the number of Dyck paths with a given number of factors (*cf.* [13]).

The Theorem 3.1 then implies the

**Corollary 3.4.** *With the notations of (3.5), the Hilbert series of  $SCov_{n,m}$  is given by*

$$F_{n,m}(t) = \frac{1-t^m}{1-t} F_n(t^m)$$

from which one deduces the close formula

$$\sum_n F_{n,m}(t) x^n = \frac{(1-t) - \sqrt{(1-t)(1-t-4t^m x(1-t^m))} - 2x(1-t^m)}{(1-t)(2t^m-1) - x(1-t^m)}.$$

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