POLYNOMIALS WITH THE HALF-PLANE PROPERTY AND RAYLEIGH MONOTONICITY

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Abstract. A polynomial $P(x)$ in n complex variables is said to have the half-plane property if $P(x) \neq 0$ whenever all the variables have positive real parts. The generating polynomial for the set of all spanning trees of a graph G is one example. Motivated by the fact that the edge set of each spanning tree of G is a basis of the graphic matroid induced by G, we show that the support of any homogeneous multiaffine polynomial with the half-plane property constitutes the set of all the bases of some matroid. Relaxing the multilinearity condition, we further show that the support is a jump system. When the polynomial has a definite parity with the half-plane property, the support is also a jump system. A related property of the spanning tree generating polynomial is the Rayleigh monotonicity property. In this paper I prove that the property holds for all the spanning tree generating polynomials using the All-Minors Matrix-Tree Theorem and Jacobi identity. I will show that the same property holds for the generating polynomial for any regular matroid and 6th-root of unity matroid by generalizing Feder and Mihail's proof [8]for the regular matroids. Open problems and a few directions for further research will also be discussed.

1. Introduction

Let $G = (V, E)$ be a connected graph. Assign to every edge $e \in E$ a weight x_e. Then by the Matrix-Tree Theorem [3], any principal cofactor of the edge-weighted Laplacian matrix $L(G)$ for G is the sum of weights of all the spanning trees of G, i.e.

$$T_G(x) = \sum_{tree T \subset E} \prod_{e \in T} x_e. \tag{1}$$

We call $T_G(x)$ the spanning tree generating polynomial of G. More generally given a collection $\mathcal{S}$ of finitely many sets, the generating polynomial for $\mathcal{S}$ is defined by $P_{\mathcal{S}}(x) = \sum_{S \in \mathcal{S}} x^S$ where x^S is used as a

Key words and phrases. Graph,set system, matroid,generating polynomial,half-plane property, jump system,Rayleigh Monotonicity.

shorthand for $\prod_{e\in S} x_e$. A polynomial $P(x)$ in n complex variables is said to have *the half-plane property* if $Re(x_i) > 0$ for all $i = 1, \ldots, n$ implies $P(x) \neq 0$. Consider the weight x_e of each edge $e \in E$ as the conductance in the electrical network defined by G. Then we have the equation $L(G)\mathbf{u} = J$, where $\mathbf{u} = \{u_i\}_{i\in V}$ is the vector of node voltages and $\mathbf{J} = \{J_i\}_{i\in V}$ is the vector of current inflows. On physical grounds, when $Re(x_e) > 0$, for all $e \in E$ the network is uniquely solvable so that the principal cofactor of $L(G)$ is nonzero [7]. Hence we have the following theorem:

Theorem 1.1. *Let G be a connected graph. Then the spanning tree polynomial T_G has the half-plane property.*

Let a and b be two distinct vertices in V. Now we consider a positive real edge weight y_e for each $e \in E$ as the conductance of e. We introduce a new directed edge e^+ from a to b and drive one ampere of current through e^+. Then by Ohm's Law, the effective resistance $\mathcal{R}_{ab}$ of $(G, \mathbf{y})$ with respect to $\{a, b\}$ is directly proportional to the potential difference between a and b. On physical grounds it is intuitively clear that the increase of y_e for any $e \in E$ cannot decrease $\mathcal{R}_{ab}$. And we can prove the following theorem, known as Rayleigh Monotonicity.

Theorem 1.2. *Let $G = (V, E)$ be a finite connected multigraph, let $\mathbf{y} := \{y_e : e \in E\}$ be positive real numbers indexed by E, and let $a, b \in V$ be distinct vertices of G. Then*

$$\frac{\partial}{\partial y_c}\mathcal{R}_{ab}(G, \mathbf{y}) \geq 0 \tag{2}$$

for every edge $c \in E$.

Algebraic simplification leads equation (2) to $\mathcal{T}_{e\bar{f}}\mathcal{T}_{\bar{e}f} - \mathcal{T}_{ef}\mathcal{T}_{\bar{e}\bar{f}} \geq 0$ for all positive real variables, where $\mathcal{T}_e$ is the generating polynomial for the set of all spanning trees which contain the edge e and $\mathcal{T}_{\bar{e}}$ is the generating polynomial for the spanning trees that do not contain e and $\mathcal{T}_{e\bar{f}}$, $\mathcal{T}_{\bar{e}\bar{f}}$, and $\mathcal{T}_{ef}$ are defined analogously. However, spanning trees of a graph have a combinatorial meaning in the theory of matroids. They are the bases of the graphic matroid $\mathcal{M}(G)$. As a generalization of the above theorem, we naturally consider the following questions :

Question 1.3. *For every matroid $\mathcal{M}$ with $\mathcal{B}$ as its set of bases, does the basis generating polynomial $P(x) := \sum_{S\in\mathcal{B}} x^S$ have the half-plane property?*

Question 1.4. *For every matroid $\mathcal{M}$ with $\mathcal{B}$ as the set of all bases of $\mathcal{M}$, does Rayleigh Monotonicity hold for any two elements $e, f \in E$, i.e. $P_{e\bar{f}}(x)P_{\bar{e}f}(x) - P(x)_{ef}P_{\bar{e}\bar{f}}(x) \geq 0$ when $x_e > 0$ for all $e \in E$?*

Question 1.5. *If the first two conjectures turn out to be false, are there special classes of matroids for which the basis generating polynomials have the half-plane property or for which Rayleigh Monotonicity holds?*

In the joint paper with J.G.Oxley, A.D.Sokal, and D.G.Wagner [7] it is shown that the Fano matroid fails to have the half-plane property. Hence the answer for $Question 1.3$ is "No". But if the generating polynomial for a given set system satisfies the half-plane property, the set system is always the set of bases of a matroid. Moreover, the theorem can be generalized by allowing the polynomial to have complex coefficients. We can also relax the multilinearity condition and prove that the support of the homogeneous polynomial constitutes a jump system. Furthermore, the support of any polynomial with definite parity and the half-plane property also constitutes a jump system. In Section 2, some basic definitions and the matroidal support theorem will be stated. Then same phase theorem and the jump system support theorem for the polynomials with the half-plane property which have definite parity will be proved. We turn to Rayleigh Monotonicity Property in Section 3, and show that regular matroids satisfy the Rayleigh Monotonicity and so do 6th-roots of unity matroids. A few open problems will be suggested in the last section.

2. Matroidal Support Theorem and Jump System Support Theorem

A *set system* $\mathcal{S}$ on the ground set E is a collection of subsets of E.

Definition 2.1. *A* matroid $\mathcal{M}$ *is an ordered pair* $(E, \mathcal{I})$ *consisting of a finite set* E *and a collection* $\mathcal{I}$ *of subsets of* E *satisfying the following three axioms:*

(A1) $\emptyset \in \mathcal{I}$.
(A2) *If* $I_1 \in \mathcal{I}$ *and* $I_2 \subseteq I_1$ *then* $I_2 \in \mathcal{I}$.
(A3) *If* I_1 *and* I_2 *are in* $\mathcal{I}$ *and* $|I_1| < |I_2|$ *then there is an element* e *of* $I_2 \setminus I_1$ *such that* $I_1 \cup \{e\} \in \mathcal{I}$.

Subsets of E in $\mathcal{I}$ are called *independent* and the maximal independent subsets of a matroid are called *the bases*.

Let $\mathcal{B}$ be the collection of bases of the matroid $\mathcal{M} = (E, \mathcal{I})$. From the definition of a matroid, we can easily deduce the following three properties of $\mathcal{B}$:

(B1) $\mathcal{B} \neq \emptyset$
(B2) $|X_1| = |X_2|$ for any $X_1, X_2 \in \mathcal{B}$
(B3) If $X_1, X_2 \in \mathcal{B}$ and $x \in X_1$ then there exists an element $y \in X_2$ so that $(X_1 \setminus x) \cup y \in \mathcal{B}$.

Conversely, if $\mathcal{B}$ is the set of certain subsets of a finite set E satisfying (B1), (B2) and (B3), then there exists a matroid $\mathcal{M} = (E, \mathcal{I})$ such that its bases are exactly the members of $\mathcal{B}$ [9].

Let $P(x)$ be the generating polynomial of a set system $\mathcal{S}$. $\mathcal{M} \setminus e$ and $\mathcal{M}/e$ are the matroids obtained from $\mathcal{M}$ by deleting and contracting e, respectively. We refer to James Oxley's book, Matroid Theory [9] for basic definitions and notations.

From now on we will use $[n] := \{1, 2, \ldots, n\}$ as the ground set of set systems. For $j \in [n]$, we denote the polynomial $\sum_{S\in\mathcal{S}\ j\notin S} \prod_{i\in S} x_i$ by $P^{\setminus j}(x)$ and $\sum_{S\in\mathcal{S},\ j\in S} \prod_{i\in S\setminus j} x_i$ by $P^{/j}(x)$.

Definition 2.2. *Let $P(x)$ be a polynomial with complex coefficients, in n complex variables. We say that $P(x)$ has* the half-plane property *if $P(x) \neq 0$ for any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ such that $Re(x_i) > 0$ for all $i = 1, 2, \ldots, n$.*

Let $P(x)$ be a homogeneous degree-r polynomial in n complex variables with complex coefficients. The following theorem gives a necessary and sufficient condition for $P(x)$ to have the half-plane property.

Theorem 2.3. (Choe,Oxley,Sokal and Wagner [7]) *Let $P(x)$ be homogeneous of degree r and let x and y be vectors in $\mathbb{R}^n$ and ζ be a real variable. Define $p_{x,y}(\zeta)$ to be the univariate polynomial $P(\zeta x + y)$ obtained by substituting $\zeta x_i + y_i$ for x_i in $P(x)$. Then the following are equivalent:*

(a) *P has the half-plane property.*
(b) *For any $x, y \in \mathbb{R}^n$ with $x, y > 0$, all the roots of $p_{x,y}(\zeta) = 0$ lie in $(-\infty, 0]$.*

Given a set system $\mathcal{S}$, we say that $\mathcal{S}$ *has the half-plane property*, if the generating polynomial for $\mathcal{S}$ has the half-plane property. In particular, a matroid $\mathcal{M}$ is said to have the half-plane property if its basis generating polynomial has the half-plane property.

Two complex numbers a and b are said to be *in the same phase* when there is a positive real number r such that $b = ra$. Let $\mathcal{B}$ be a set of r-subsets of a finite set $[n]$. When a polynomial $P_{\mathcal{B}}(x)$ has the half-plane property, the coefficients $\{a_S\}_{S\in\mathcal{B}}$ are in the same phase.

Theorem 2.4. *If a homogeneous multilinear polynomial $P(x) = \sum_{S\in\mathcal{B}} a_S x^S$ of degree r with complex coefficients has the half-plane property then all the coefficients have the same phase.*

Using the same phase theorem, we can prove the following matroidal support theorem.

Theorem 2.5. *If a homogeneous multilinear polynomial $P(x) = \sum_{S\in\mathcal{B}} a_S x^S$ of degree r with complex coefficients has the half-plane property then there exists a matroid $\mathcal{M}$ such that the set of its bases is exactly $\mathcal{B}$.*

Let $\mathcal{S}_r$ be the set of functions, $\alpha : [n] \to \mathbb{N}$ such that $\sum_{i\in[n]} \alpha(i) = r$. Given a homogeneous degree-r polynomial $P(x) = \sum_{\alpha\in\mathcal{S}_r} a_\alpha x^\alpha$, define *Supp* $P(x)$ to be the subset of $\mathcal{S}_r$ such that $\{\alpha \in \mathcal{S}_r : a_\alpha \neq 0\}$. Denote the function $min\{\alpha, \beta\}$ for $\alpha, \beta \in \mathcal{S}_r$ by $\alpha \wedge \beta$. Let the function $\alpha_{\setminus e}$ be defined to have the same value as α except for $\alpha_{\setminus e}(e) := \alpha(e) - 1$. Similarly, define $\alpha_{\cup f}$ to differ from α only at f, i.e. $\alpha_{\cup f}(f) := \alpha(f) + 1$. The following generalizes Theorem 2.4.

Theorem 2.6. *Let $P(x) = \sum_{\alpha\in\mathcal{S}_r} a_\alpha x^\alpha$ be a homogeneous degree-r polynomial with the half-plane property with complex coefficients. Let $\alpha, \beta \in Supp\ P(x)$. Then for any $e \in [n]$ such that $\alpha(e) > \beta(e)$, there exists $f \in [n]$ with $\alpha(f) < \beta(f)$ satisfying $\alpha_{\setminus e\cup f} \in SuppP(x)$.*

To prove the theorem, we need a generalized same-phase theorem.

Theorem 2.7. (Choe,Oxley,Sokal and Wagner [7]) *Let $P(x) = \sum_\alpha a_\alpha x^\alpha$ be a polynomial in n complex variables that is homogeneous of degree r. If P has the half-plane property, then all the nonzero coefficients a_α have the same phase.*

For $x, y \in \mathbb{Z}^{[n]}$, let $\|x\| = \sum_{i\in[n]} |x_i|$ and $d(x, y) = \|x - y\|$. Now we extend the meaning of the bases in a matroid and define the jump system as follows. This was first introduced by Bouchet and Cunningham.

Definition 2.8. (Bouchet and Cunningham [2]) *For $x, y \in \mathbb{Z}^{[n]}$,* a step from x to y *is a vector $u \in \mathbb{Z}^{[n]}$ such that $\|u\| = 1$ and $d(x + u, y) = d(x, y) - 1$. Let $St(x, y)$ be the set of steps from x to y. A pair $([n], \mathcal{J})$ is called a* jump system *if it satisfies the following* 2-step axiom*:*
2-step axiom *If $x, y \in \mathcal{J}, u \in St(x, y)$, and $x + u \notin \mathcal{J}$, then there exists $v \in St(x + u, y)$ with $x + u + v \in \mathcal{J}$.*

Note that we can consider the members in $\mathcal{S}_r$ as vectors in $\mathbb{N}^{[n]}$.

Proposition 2.9. *Let $P(x) = \sum_{\alpha\in\mathcal{S}_r} a_\alpha x^\alpha$ be a homogeneous degree-r polynomial in n complex variables where $a_\alpha \in \mathbb{C}$ and $n \geq 2$. If $P(x)$ has the half-plane property, then $([n], SuppP(x))$ is a jump system.*

Now we generalize the same phase theorem and the jump system support theorem to the polynomials all of whose terms have degrees with the same parity. We say those polynomials have definite parity. To prove the main theorem, we need the generalized same phase theorem.

Theorem 2.10. (Wagner [6]) *Let $P(x) = \sum_{\alpha \in \mathcal{S}} a_\alpha x^\alpha$ be a polynomial in n complex variables with definite parity, where $a_\alpha \in \mathbb{C}$ and $n \geq 2$. If $P(x)$ has the half-plane property, then all the coefficients have the same phase.*

Here we state a more generalized form of the support theorem:

Theorem 2.11. (Choe [6]) *Let $P(x) = \sum_{\alpha \in \mathcal{S}} a_\alpha x^\alpha$ be a polynomial in n complex variables with definite parity, where $a_\alpha \in \mathbb{C}$ and $n \geq 2$. If $P(x)$ has the half-plane property, then $([n], SuppP(x))$ is a jump system.*

It seems easier to deal with multilinear polynomials than the polynomials which are not multilinear. To prove Theorem 2.11, we use a nice tool called *polarization* with which we convert the polynomial $P(x)$ in the theorem into a multilinear polynomial $\mathcal{P}(P)$ with the half-plane property and definite parity.([6]) We then show the following properties.

Lemma 2.12. (Choe [6]) *Let $P(x)$ be a polynomial in n complex variables. Then $P(x)$ has the half-plane property if and only if $\mathcal{P}(P)$ has the half-plane property.*

Thus the following theorem implies Theorem 2.11.

Theorem 2.13. (Choe [6]) *Let $P(x) = \sum_{U \subset E} a_U x^U$ be a multilinear homogeneous polynomial in n complex variables with definite parity, where $a_U \in \mathbb{C}$ and $n \geq 2$. If $P(x)$ has the half-plane property , then $(E,\ Supp\ P(x))$ is a jump system.*

The above theorem can be proven by induction on the degree r of $P(x)$. For the first part of the induction step, we use an extensive case analysis and then for larger cases, we can use the induction hypothesis along with the proven facts from the case analysis.

3. Rayleigh Monotonicity of $\sqrt[6]{1}$- matroids

For graphs, Rayleigh Monotonicity can be proven by using the Jacobi Identity [1] and Chaiken's All Minors Matrix-Tree Theorem [4]. Since the set of spanning trees of a graph G is the set of bases in the graphic matroid $\mathcal{M}(G)$, one natural attempt can be to prove the Rayleigh Monotonicity Property for some other collection of matroids, for example, regular matroids. Feder and Mihail [8] Showed that Rayleigh

Monotonicity holds when all the variables are set to 1. In this section, the Rayleigh Monotonicity Property for regular matroids will be shown for all positive real variables by adopting the idea which was used by Feder and Mihail [8] in proving the negative correlation of regular matroids from the probabilistic point of view. Furthermore, Rayleigh Monotonicity will be proven for all $\sqrt[6]{1}$-matoids.

Here we need circuit axioms to define a matroid, which was first used by Tutte and Whitney.

(C1) If $C_1,\ C_2$ are distinct circuits then $C_1 \nsubseteq C_2$.
(C2) If $C_1,\ C_2$ circuits and $z \in C_1 \cap C_2$ there exists a circuit C_3 such that $C_3 \subset (C_1 \cup C_2) \setminus z$.
(C3) If C_1, C_2 are distinct members of $\mathcal{C}$ and $y \in C_1 \setminus C_2$, there exists $C_3 \in \mathcal{C}$ such that $y \in C_3 \subset (C_1 \cup C_2) \setminus x$.

(C1) and (C2) axiomatise a matroid by its circuits. (C3) together with (C1) are also axioms which define a matroid.

Now we'll introduce the algebra of chain groups and its connection with the circuits of a matroid [11]. Let R be a commutative integral domain and S be a finite set. A *chain* on S over R is a map $f : S \to R$. The subset of S consisting of the elements at which f has nonzero value is called the support of f and denoted by $\|f\|$. A *chain group* on S over R is a collection of chains on S over R which is closed under the two operations, sum and product.We denote the collection of chains by $\mathcal{C}(S, R)$.

Theorem 3.1. (Welsh [11]) *Let N be a chain group on S over R and let $\mathcal{D}(N)$ be the set of supports of the chains of N. Then $\mathcal{D}(N)$ is the collection of dependent sets of a matroid on S.*

We call the above matroid obtained from a chain group N *the matroid of the chain group* N and denote it by $\mathcal{M}(N)$.

Theorem 3.2. (Welsh [11]) *A matroid $\mathcal{M}$ on S is isomorphic to the matroid $\mathcal{M}(N)$ of a chain group N over a field F if and only if $\mathcal{M}$ is representable over F. Moreover if $\mathcal{M} \cong \mathcal{M}(N)$ then $\mathcal{M}$ is isomorphic to the matroid induced on the quotient space $\mathcal{C}(S, F)/N$ by linear independence.*

Now we can think of the relation between the matroid of the chain group and the orthogonal complement of the chain group.

Proposition 3.3. (Welsh [11]) *Let A be an $m \times n$ matrix over a field F and $\mathcal{M} = \mathcal{M}[A]$. Then the set of cocircuits of $\mathcal{M}$ coincides with the set of minimal nonempty supports of vectors from the row space of A.*

Theorem 3.4. (Welsh [11]) *Let $\mathcal{M} = \mathcal{M}(N)$ be the matroid of the chain group $N \subset \mathcal{M}(S, \mathbb{R})$. Then $\mathcal{M}^* = \mathcal{M}(N^{\perp})$ where $N^{\perp}$ is the orthogonal complement of N with respect to the bilinear map $b : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ defined by*

$$b(f, g) = \sum_{e \in S} f(e)g(e) \tag{3}$$

However, for $\mathcal{M}(S, \mathbb{C})$ where $\mathbb{C}$ is the complex field, b is defined by

$$b(f, g) = \sum_{e \in S} \overline{f(e)}g(e). \tag{4}$$

Hence for every pair of a circuit and a cocircuit of a matroid which is representable over a field, there is a corresponding pair of chains f and g over the field such that the inner product of f and g is zero. From this fact, all regular matroids are orientable. That is, for each circuit C and each cocircuit D of a given matroid $\mathcal{M}$, we can assign a value from $\{1, -1\}$ to $C(e)$ for all $e \in C$ and to $D(f)$ for each $f \in D$ and zero to $C(g), D(h)$ for all $g \notin C$, $h \notin D$, respectively, so that the inner product of the vectors C and D is zero: that is, $\sum_{g \in S} C(g)D(g) = 0$.

An analogous definition to the regular matroids can be made over the complex field $\mathbb{C}$ for the class of $\sqrt[6]{1}$-matroids. When every nonzero subdeterminant of a matrix M is a complex sixth root of unity, we call M a $\sqrt[6]{1}$-matrix. Any matroid that can be represented over $\mathbb{C}$ by a $\sqrt[6]{1}$-matrix is called a $\sqrt[6]{1}$-matroid. A matroid B is obtained from A by a *proper scaling* of A if B is obtained by multiplying some row or column of A by a member of the set $\{\pm 1, \pm r, \pm(r+1)\}$ where r denotes a complex root of $\alpha^2 - \alpha + 1$.

Proposition 3.5 (Whittle [12]). *Let A be a $\sqrt[6]{1}$-matrix and B be a matrix over $\mathbb{C}$.*

1. *If B is obtained from A by a sequence of proper scalings, then B is a $\sqrt[6]{1}$-matrix.*
2. *If B is obtained from A by a sequence of pivots, then B is a $\sqrt[6]{1}$-matrix.*

In a similar way as regular matroids and by using the above properties, we can assign a sixth root of unity to $C(g)$ for each $g \in C$ where C is a circuit of $\mathcal{M}$.

Consider the independent sets N which are obtained from the bases by removing one element in the basis. The complement of N is dependent in $\mathcal{M}^*$ and contains a basis of the dual matroid. Hence it contains a cocircuit D_N. We call N a *near basis* and D_N the *cut* of N. For any spanning set U whose cardinality is one more than that of a basis, there is a unique cycle C_U contained in U. U is called a *unicycle*. Let's assume that a $\sqrt[6]{1}$-matroid is given with a set of vectors for the circuits that are defined as above. Let e and f be in the ground set of the matroid. Without loss of generality we may assume that for every circuit C and cocircuit D that contain e, $C(e) = -1$ and $D(e) = 1$. Especially when $U = N \cup \{e, f\}$,we get the following equation.

$$\sum_g \overline{C_U(g)} D_N(g) = \overline{C_U(e)} D_N(e) + \overline{C_U(f)} D_N(f) = 0.$$

Thus we have $\overline{C_U(f)} D_N(f) = 1$ and since ${C_U}^{-1}(f) = \overline{C_U(f)}$, we get $C_U(f) = D_N(f)$. Let $\mathcal{T}$ be the generating polynomial for the set of bases in a matroid $\mathcal{M}$, that is, $\mathcal{T} = \sum_{B \in \mathcal{B}(\mathcal{M})} x^B$. For a pair of distinct elements e and f, we define $\triangle_{ef}$ by $\sum_N D_N(e) D_N(f) x^N$ where the sum is over all near-bases of $\mathcal{M}$. Combining all the results we've obtained, we can prove the following:

Theorem 3.6. (Choe [6]) *For a $\sqrt[6]{1}$-matroid $\mathcal{M}$, the following equation holds for any pair e, f of elements in $\mathcal{M}$.*

$$\mathcal{T}_{e\overline{f}} \mathcal{T}_{\overline{e}f} - \mathcal{T}_{ef} \mathcal{T}_{\overline{e}\overline{f}} = x_e x_f \triangle_{ef} \overline{\triangle_{ef}} \tag{5}$$

Proof. For every circuit and cocircuit of $\mathcal{M}$, we assign chains over $\mathbb{C}$, $C : E(C) \to \{r \in \mathbb{C} : r^6 = 1\}$,$D : E(D) \to \{r \in \mathbb{C} : r^6 = 1\}$,respectively so that we always have $\sum_g \overline{C(g)} D(g) = 0$. Hence $\sum_{g \in E g \neq e} \overline{C(g)} D(g) + \overline{C(e)} D(e) = 0$, that is,

$$\sum_{g \in E g \neq e} \overline{C(g)} D(g) = -\overline{C(e)} D(e) = 1.$$

Since $\mathcal{T}_e = \mathcal{T}_{ef} + \mathcal{T}_{e\overline{f}}$, $\mathcal{T}_e \mathcal{T}_{\overline{e}f} - \mathcal{T}_{\overline{e}} \mathcal{T}_{ef} = \mathcal{T}_{\overline{e}f} \mathcal{T}_{e\overline{f}} - \mathcal{T}_{ef} \mathcal{T}_{\overline{ef}}$. Let (T_1, T_2) be a pair of bases in $\mathcal{T}_e \times \mathcal{T}_{\overline{e}f}$. So it suffices to show that $\mathcal{T}_e \mathcal{T}_{\overline{e}f} - \mathcal{T}_{\overline{e}} \mathcal{T}_{ef} = x_e x_f \triangle_{ef} \overline{\triangle_{ef}}$. For each element $g \in E$, we define the weight of the pair (T_1, T_2) at g by $\overline{D_{T_1 \setminus e}(g)} C_{T_2 \cup e}(g) x_{T_1} x_{T_2}$. Note that the weight is zero if g is not in the cocircuit $D_{T_1 \setminus e}$ induced by the near-basis $T_1 \setminus e$ or not in the circuit $C_{T_2 \cup e}$ induced by the unicircuit $T_2 \cup e$. Let $g (\neq f)$ be contained in both $D_{T_1 \setminus e}$ and $C_{T_2 \cup e}$. Then $T_1 \setminus e \cup \{g\}$ is a basis in $\mathcal{T}_{\overline{e}}$ and we name the basis T_3. $T_2 \setminus g \cup \{e\}$ is in $\mathcal{T}_{ef}$ and we call it T_4. Now, $T_1 \setminus e = (T_3 \cup e) \setminus \{e, g\}$ and hence by (1) we have $D_{T_1 \setminus e}(g) = C_{T_3 \cup e}(g)$. Analogously, we have $C_{T_2 \cup e}(g) = D_{T_4 \setminus e}(g)$. Therefore, $\overline{D_{T_1 \setminus e}(g)} C_{T_2 \cup e}(g) x_{T_1} x_{T_2} = \overline{C_{T_3 \cup e}(g)} D_{T_4 \setminus e}(g) x_{T3 \setminus g \cup e} x_{T_4 \setminus e \cup g} =$

$\overline{C_{T_3\cup e}(g)}D_{T_4\backslash e}(g)x_{T_3}x_{T_4}$.

$$\begin{aligned}
\mathcal{T}_e\mathcal{T}_{\overline{e}f} = & \sum_{(T_1,T_2)\in\mathcal{T}_e\times\mathcal{T}_{\overline{e}f}} (\sum_{g\neq e,f} \overline{D_{T_1\backslash e}(g)}C_{T_2\cup e}(g))\ x^{T_1}x^{T_2} \\
& + \sum_{(T_1,T_2)\in\mathcal{T}_e\times\mathcal{T}_{\overline{e}f}} \overline{D_{T_1\backslash e}(f)}C_{T_2\cup e}(f)\ x^{T_1}x^{T_2} \\
= & \sum_{(T_3,T_4)\in\mathcal{T}_{\overline{e}}\times\mathcal{T}_{ef}} (\sum_{g\neq e,f} \overline{C_{T_3\cup e}(g)}D_{T_4\backslash e}(g))\ x^{T_3}x^{T_4} \\
& + (\sum_{T_3\in\mathcal{T}_e} \overline{D_{T_1\backslash e}(f)}x^{T_3})(\sum_{T_4\in\mathcal{T}_{\overline{e}}} C_{T_2\cup e}(f)x^{T_4}) \\
= & \sum_{(T_3,T_4)\in\mathcal{T}_{\overline{e}}\times\mathcal{T}_{ef}} x^{T_3}x^{T_4} + (\sum_{N:e\in D_N} \overline{D_N(f)}x^{N\cup e})(\sum_{U:e\in C_U} C_U(f)x^{U\backslash e}) \\
= & \quad \mathcal{T}_{\overline{e}}\mathcal{T}_{ef} + x_e x_f \overline{\triangle_{ef}}\triangle_{ef}
\end{aligned}$$

Therefore, the result follows.

$$\mathcal{T}_{\overline{e}f}\mathcal{T}_{e\overline{f}} - \mathcal{T}_{ef}\mathcal{T}_{\overline{ef}} = x_e x_f \overline{\triangle_{ef}}\triangle_{ef}.$$

□

Corollary 3.7. (Choe [6]) *For a regular matroid $\mathcal{M}$, the following holds for every pair (e,f) of elements in $E(\mathcal{M})$.*

$$\mathcal{T}_{e\overline{f}}\mathcal{T}_{\overline{e}f} - \mathcal{T}_{ef}\mathcal{T}_{\overline{e}\overline{f}} = x_e x_f {\triangle_{ef}}^2 \tag{6}$$

If the generating polynomial $\mathcal{T}$ of a matroid $\mathcal{M}$ on E satisfies the inequality (6) for all nonnegative real variables, we call $\mathcal{M}$ a *Rayleigh matroid*. All uniform matroids and matching matroids are proven to be Rayleigh matroids.([6]) The collection of Rayleigh matroids are closed under taking minors, duality, direct sums, and 2-sums.([6]) We can also show that Rayleigh monotonicity holds for a pair of two disjoint subsets of E as well as a pair of distinct elements.([6])

Some of the interesting open problems are as follows:

- finding sufficient conditions for a polynomial with the half-plane property
- finding operations in polynomials which preserve the half-plane property
- constructing a collection of polynomials with the half-plane property using the given collection of polynomials with the half-plane property

- generalizing the All Minors Matrix-Tree Theorem for some collection of matroids
- finding minor minimal binary Rayleigh matroids which do not have the half-plane property.

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