THE NUMBER OF SPANNING TREES IN GRAPHS RELATED TO CIRCULANT GRAPHS (EXTENDED ABSTRACT)*

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Abstract. In this paper we consider the number of spanning trees in the complete graph with circulant graphs deleted (added) from (to) it. By using the properties of *Chebyshev* polynomials, we derive closed formulae for the number of spanning trees in many classes of such graphs.

Résumé. Dans cet article, nous considérons le nombre d'arbres couvrants d'un graphe complet auquel on ajoute ou on retranche des graphes circulants. En utilisant des propriétés des polynômes de *Tchebicheff,* nous obtenons des formules closes pour le nombre d'arbres couvrants d'un grand nombre de classes de tels graphes.

1. Introduction

An undirected graph G is a pair (V, E), in which V is the vertex set and $E \subseteq V \times V$ is the edge set. In a graph, a *(self-)loop* is an edge joining a vertex to itself and *multiple edges* are several edges joining the same two vertices. All graphs considered in this paper are finite, and undirected with self-loops and multiple edges permitted.

The complete graph on n vertices, denoted by K_n, has one edge between each pair of distinct vertices. Let S be a subset of the edge set of K_n (or S be a subgraph of K_n). We denote by $K_n - S$ the graph remaining when all edges in S are removed from K_n. If S is a subgraph of K_n, $K_n - S$ is called the *complement* graph of S in K_n, and also denoted as $\overline{S}$. For an edge set S, we denote by $K_n + S$ the graph K_n with all edges in S added to it; if S is nonempty then $K_n + S$ is a graph containing some multiple edges. In Figure 1 we give examples of two graphs, one is the graph $K_6 - C_4$ which is the complete graph K_6 with a cycle of 4 edges C_4 deleted from it; another is the graph $K_6 + C_4$ which is K_6 with a cycle C_4 added to it. In the graph $K_6 - C_4$, the dashed lines are edges deleted; in the graph $K_6 + C_4$, the dashed lines are edges added.

A spanning tree in a graph G is a tree which has the same vertex set with G. The number of spanning trees in a graph (network) G is an important quantity to measure the reliability of G [8]. For graph G, the number of spanning trees in G is denoted as $T(G)$. The problem of calculating $T(K_n - S)$ has already been studied for many different types of S. The initial work seems to have been by *Weinberg* [18] who gave formulae for $T(K_n - S)$ when all edges in S are not adjacent or are adjacent at one vertex. Subsequently, in a series of papers [1, 2, 3, 4], *Bedrosian* extended this to show how to calculate $T(K_n - S)$ when all edges in S are not adjacent or adjacent at one vertex, or form a path, a cycle, a complete graph, or are some combination of these configurations. *Weinberg*'s results have also been generalized in [15]. Closed formulae also exist for the cases where S is a star [14], a complete K-partite graph [16], a multi-star [13, 19], and so on. The number of spanning trees in

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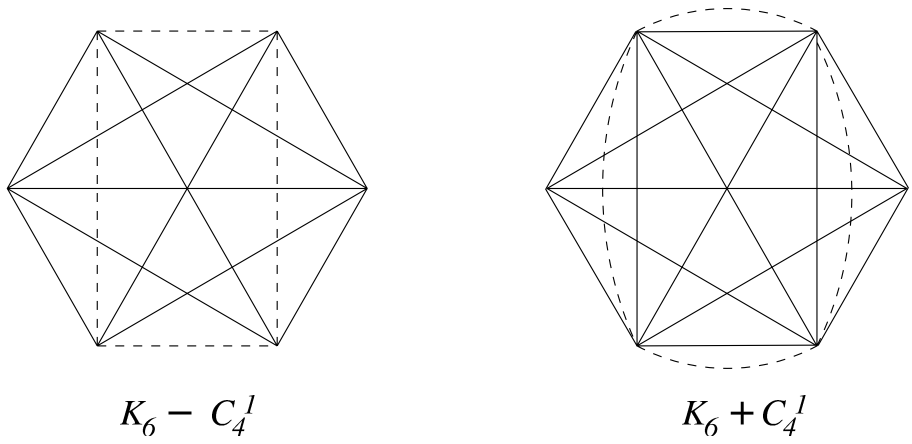

FIGURE 1. Two examples of graphs.

the complement graph is investigated in [9, 11] when the graph with maximal number of spanning trees is studied. The formulae for the number of spanning trees in the complement graphs of a disjoint union of cycles or paths are given in generic forms in [9]. Not as much is known about $T(K_n + S)$; *Bedrosian* [2] considered it for some simple configurations S, i.e., all edges in S form a cycle, complete graph, or $|S|$ is quite small but much more does not seem to be known.

In this paper we describe how to calculate $T(K_n \pm S)$ where S is a circulant graph. Let $1 \leq s_1 < s_2 < \cdots < s_k$. The *undirected circulant graph*, $C_n^{s_1,s_2,\cdots,s_k}$, has n vertices labeled $0,\ 1,\ 2,\ \cdots,\ n-1$, with each vertex i $(0 \leq i \leq n-1)$ adjacent to $2k$ vertices $i \pm s_1,\ i \pm s_2,\ \cdots,\ i \pm s_k \bmod n$. Note that as a special case, C_n^1 is simply the n node cycle. More generally, if $(n,\ s) = 1$ then C_n^s is the n node cycle while if $(n,\ s) = d > 1$ then C_n^s is the disjoint union of d cycles $C_{n/d}^1$.

The number of spanning trees in circulant graphs has been well-studied (see [20] for exposition). But, other than for the graphs $K_n - S$ as mentioned above where S is a cycle or a disjoint union of cycles, there does not seem to have been any other work about the number of spanning trees in a complete graph with a circulant graph deleted (added) from (to) it, especially, no closed formulae for the number of spanning trees in a complete graph with a arbitrary circulant graphs deleted (added) from (to) it. This is the problem to be considered in this paper.

In Section 2 we state some lemmas about $T(G)$ and review some properties of *Chebyshev* polynomials. In Section 3 we use these lemmas and properties to derive a series of formulae for the numbers of spanning trees in the complete graphs with circulant graphs deleted (added) from (to) them. Our approach is to first start by developing a new approach to deriving a closed form for $T(K_n - C_m^s)$, where C_m^s is a cycle or union of cycles (a closed form for this was previously derived using different techniques in [9]). We then continue by showing that it is easy to generalize this approach to getting a formula for $T(K_n \pm C_m^{s_1,s_2,\cdots,s_k})$. In the case that all of the $s_i \leq 4$ we will actually be able to derive a simple closed form function $g(n, m; s_1, s_2, \cdots, s_k) = T(K_n \pm C_m^{s_1,s_2,\cdots,s_k})$ of $n,\ m$. We conclude in Section 4 by describing extensions and limitations of our technique.

2. Basic concepts and lemmas

For graph G, we denote by $A(G)$ or A the adjacency matrix of $G = (V, E)$. If $V = \{v_1,\ v_2,\ \cdots,\ v_n\}$, A is the $n \times n$ matrix with $a_{i,j}$ being the number of edges connecting

v_i and v_j. The degree, d_i, of vertex v_i is $\sum_{j=1}^{n} a_{i,j}$, the number of edges adjacent to v_i. Let B denote the diagonal matrix with $\{d_1,\ d_2,\ \cdots,\ d_n\}$ as diagonal entries. The classic result known as the *Matrix Tree Theorem* [12] states that, the *Kirchhoff* matrix H defined as $H = B - A$ has all its co-factors[1] equal to $T(G)$. For example, the *Kirchhoff* matrix H of the graph $K_6 - C_4$ shown in Figure 1 is

$$H = \begin{pmatrix} 3 & 0 & -1 & -1 & 0 & -1 \\ 0 & 3 & -1 & 0 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & 0 & -1 & 3 & 0 & -1 \\ 0 & -1 & -1 & 0 & 3 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 \end{pmatrix},$$

all its co-factors are 192 which is the number of spanning trees in $K_6 - C_4$.

The number of spanning trees in graph G also can be calculated from the eigenvalues of the *Kirchhoff* matrix H. All eigenvalues of a real symmetric matrix are real. Even more, by basic knowledge of linear algebra, all eigenvalues of the *Kirchhoff* matrix H of a graph with n nodes are non-negative, and 0 is one of its eigenvalues because all its row vectors add up to 0 vector. So we can let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n (= 0)$ denote all eigenvalues of H. *Kel'mans* and *Chelnokov* [11] have shown that

$$T(G) = \frac{1}{n} \prod_{j=1}^{n-1} \mu_j. \tag{1}$$

In order to use this equation we will need to know the eigenvalues of the *Kirchhoff* matrices of the appropriate graphs. This will require the following lemmas:

Lemma 1. (Biggs [5], Page 16) *The* Kirchhoff *matrix of the circulant graph* $C_n^{s_1, s_2, \cdots, s_k}$ *has* n *eigenvalues as* $2k - \varepsilon^{-s_1 j} - \cdots - \varepsilon^{-s_k j} - \varepsilon^{s_1 j} - \cdots - \varepsilon^{s_k j}$, $1 \leq j \leq n-1$ *and* 0, *where* ε^{-j} *is the conjugate of* ε^j, $\varepsilon = e^{\frac{2\pi i}{n}}$.

Lemma 2. ([11]) *Let* G *be a graph with* n *vertices and* $\overline{G}$ *the complement graph of* G *in* K_n. *If the* Kirchhoff *matrix of* G *has eigenvalues* $\mu_1,\ \mu_2,\ \cdots,\ \mu_{n-1}$ *and* 0, *then the* Kirchhoff *matrix of* $\overline{G}$ *has eigenvalues* $n - \mu_1,\ n - \mu_2,\ \cdots,\ n - \mu_{n-1}$ *and* 0.

Using basic multilinear algebra the following lemma can be proven in a way similar to that of the proof of Lemma 2.

Lemma 3. G *is a graph with same vertex set as* K_n. *If the* Kirchhoff *matrix of* G *has eigenvalues* $\mu_1,\ \mu_2,\ \cdots,\ \mu_{n-1}$ *and* 0, *then the* Kirchhoff *matrix of* $K_n + G$ *has eigenvalues* $n + \mu_1,\ n + \mu_2,\ \cdots,\ n + \mu_{n-1}$ *and* 0.

Let $G_1 = (V_1,\ E_1)$ and $G_2 = (V_2,\ E_2)$ be two graphs with disjoint vertex sets. The join $G = G_1 \bigoplus G_2$ is defined as the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{uv | u \in V_1,\ v \in V_2\}$ [7]. (Please note that in this paper we use "$\bigoplus$" to denote the join graph instead of "+" as used in some other references. This is because we are already using "+" to denote the graph that results by adding edges to some other graph.) The following lemma describes the relation of the eigenvalues of the *Kirchhoff* matrix of the join graph and the eigenvalues of *Kirchhoff* matrices of the original graphs.

Lemma 4. ([10, 11]) *If the* Kirchhoff *matrix of graph* G_1 *with* n *vertices has eigenvalues* $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n (= 0)$ *and that of graph* G_2 *with* m *vertices has eigenvalues* $\mu_1,\ \mu_2,\ \cdots,$

[1]The (i, j)th cofactor of A is the determinant of the $(n-1) \times (n-1)$ matrix that results from deleting the ith row and jth column from A.

$\mu_m(=0)$, *then the* Kirchhoff *matrix of the join* $G_1 \bigoplus G_2$ *has eigenvalues* $m+n$, λ_1+m, $\cdots$, $\lambda_{n-1}+m$ *and* μ_1+n, $\cdots$, $\mu_{m-1}+n$, 0.

Let $C_{m_1}^{s_{1_1},s_{2_1},\cdots,s_{k_1}}$, $C_{m_2}^{s_{1_2},s_{2_2},\cdots,s_{k_2}}$, $\cdots$, $C_{m_l}^{s_{1_l},s_{2_l},\cdots,s_{k_l}}$ be a collection of circulant graphs, and $\bigcup_{u=1}^{l} C_{m_u}^{s_{1_u},s_{2_u},\cdots,s_{k_u}}$ be their disjoint union. For each u, $1 \le u \le l$, suppose $m_u > 2s_{k_u}$ and let $\overline{C_{m_u}^{s_{1_u},s_{2_u},\cdots,s_{k_u}}}$ be the complement graph of $C_{m_u}^{s_{1_u},s_{2_u},\cdots,s_{k_u}}$ in K_{m_u}. Note that, for any n, $n \ge \sum_{u=1}^{l} m_u$,

$$\begin{aligned} &K_n - \bigcup_{u=1}^{l} C_{m_u}^{s_{1_u},s_{2_u},\cdots,s_{k_u}} \\ &= \left(K_{n-\sum_{u=1}^{l} m_u}\right) \bigoplus \left(K_{m_1} - C_{m_1}^{s_{1_1},s_{2_1},\cdots,s_{k_1}}\right) \bigoplus \cdots \bigoplus \left(K_{m_l} - C_{m_l}^{s_{1_l},s_{2_l},\cdots,s_{k_l}}\right) \\ &= \left(K_{n-\sum_{u=1}^{l} m_u}\right) \bigoplus \overline{C_{m_1}^{s_{1_1},s_{2_1},\cdots,s_{k_1}}} \bigoplus \cdots \bigoplus \overline{C_{m_l}^{s_{1_l},s_{2_l},\cdots,s_{k_l}}}. \end{aligned}$$

So, by Lemma 1, Lemma 2, Lemma 4 and (1), we have the following result.

Corollary 1. *For* $n \ge \sum_{u=1}^{l} m_u$ *and for each* u, $1 \le u \le l$, $m_u > 2s_{k_u}$,

$$\begin{aligned} &T(K_n - \bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\cdots,s_{ku}}) \\ &= n^{n-\sum_{u=1}^{l} m_u + l-2} \prod_{u=1}^{l} \prod_{j=1}^{m_u-1} (n - 2k_u + \varepsilon_u^{-s_{1u}j} + \cdots + \varepsilon_u^{-s_{ku}j} + \varepsilon_u^{s_{1u}j} + \cdots + \varepsilon_u^{s_{ku}j}), \end{aligned}$$

where $\varepsilon_u = e^{\frac{2\pi i}{m_u}}$, *for each* u, $1 \le u \le l$.

In a similar fashion, the following corollary can be derived from Lemma 1, Lemma 3, Lemma 4 and (1):

Corollary 2. *For* $n \ge \sum_{u=1}^{l} m_u$,

$$\begin{aligned} &T(K_n + \bigcup_{u=1}^{l} C_{m_u}^{s_{1u},s_{2u},\cdots,s_{ku}}) \\ &= n^{n-\sum_{u=1}^{l} m_u + l-2} \prod_{u=1}^{l} \prod_{j=1}^{m_u-1} (n + 2k_u - \varepsilon_u^{-s_{1u}j} - \cdots - \varepsilon_u^{-s_{ku}j} - \varepsilon_u^{s_{1u}j} - \cdots - \varepsilon_u^{s_{ku}j}), \end{aligned}$$

where $\varepsilon_u = e^{\frac{2\pi i}{m_u}}$, *for each* u, $1 \le u \le l$.

In order to evaluate these products we will show how to transform them into functions of *Chebyshev* polynomials; to do this we will need some special properties of these polynomials [6, 17]. The properties, listed below, are taken from [6] which used them to derive the number of spanning trees of some special classes of graphs, e.g., wheels, fans, ladders, *Moebius* ladders, squares of cycles and complete prisms.

The properties of the *Chebyshev* polynomials [6, 17] are used also in this paper. The following definitions and derivations are from [6]. For positive integer m, the *Chebyshev* polynomials of the first kind are defined by

$$T_m(x) = \cos(m \arccos x). \tag{2}$$

The *Chebyshev* polynomials of the second kind are defined by

$$U_{m-1}(x) = \frac{1}{m}\frac{d}{dx}T_m(x) = \frac{\sin(m \arccos x)}{\sin(\arccos x)}. \tag{3}$$

It's easily verified that

$$U_m(x) - 2xU_{m-1}(x) + U_{m-2}(x) = 0. \tag{4}$$

Solving this recursion by using standard methods, the following explicit formula is obtained

$$U_m(x) = \frac{1}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^{m+1} - (x-\sqrt{x^2-1})^{m+1}]. \tag{5}$$

The definition of $U_m(x)$ easily yields its zeros and it's verified that

$$U_{m-1}(x) = 2^{m-1}\prod_{j=1}^{m-1}[x - \cos(j\pi/m)]. \tag{6}$$

Further one notes that

$$U_{m-1}(-x) = (-1)^{m-1}U_{m-1}(x). \tag{7}$$

These two results yield another formula for $U_m(x)$

$$U_{m-1}^2(x) = 4^{m-1}\prod_{j=1}^{m-1}[x^2 - \cos^2(j\pi/m)]. \tag{8}$$

3. Results

We are now ready to derive the main result of this paper, a way to calculate $T(K_n \pm S)$ when S is a circulant graph. We start by assuming that $S = C_m^s$. As previously noted, if $(m,s) = 1$ this is just the m-cycle and if $(m,s) = d > 1$ this is the disjoint union of d cycles, each of length m/d.

Before proceeding we note that *Gilbert* and *Myrvold* [9] already gave a formula for the number of spanning trees in the graph $K_n - S$ where S is the disjoint union of cycles. The following theorem can actually be derived from *Gilbert* and *Myrvold*'s formula. The proof here is new, though; we derive it since it provides a 'pure' way of illustrating the techniques we will use later.

Theorem 5. *For $n \geq m > 2s$, if $(m,\ s) = d$, then*

$$T(K_n - C_m^s) = n^{n-m-2}\left[\left(\sqrt{\frac{n}{4}}+\sqrt{\frac{n-4}{4}}\right)^{m/d} - \left(-\sqrt{\frac{n}{4}}+\sqrt{\frac{n-4}{4}}\right)^{m/d}\right]^{2d}.$$

Proof. Let $\varepsilon_1 = e^{\frac{2d\pi i}{m}}$. If $(m,\ s) = d$, then C_m^s is the disjoint union of d cycles $C_{m/d}^1$. So, by Corollary 1, we have

$$\begin{aligned}
T(K_n - C_m^s) &= T(K_n - \bigcup_{u=1}^{d} C_{m/d}^1) \\
&= n^{n-m+d-2} \prod_{u=1}^{d} \prod_{j=1}^{\frac{m}{d}-1} (n - 2 + \varepsilon_1^{-j} + \varepsilon_1^{j}) \\
&= n^{n-m+d-2} \prod_{u=1}^{d} \prod_{j=1}^{\frac{m}{d}-1} (n - 2 + 2\cos(2dj\pi/m)) \\
&= n^{n-m+d-2} \prod_{u=1}^{d} \left[(-4)^{\frac{m}{d}-1} \prod_{j=1}^{\frac{m}{d}-1} \left(\frac{-n+4}{4} - \cos^2(dj\pi/m) \right) \right],
\end{aligned}$$

where we are using the fact that $1 + \cos(2x) = 2\cos^2 x$.

Applying the formulas (8) and then (5) yields the required

$$\begin{aligned}
T(K_n - C_m^s) &= n^{n-m+d-2} \prod_{u=1}^{d} \left[(-1)^{\frac{m}{d}-1} U^2_{\frac{m}{d}-1} \left(\sqrt{\frac{-n+4}{4}} \right) \right] \\
&= n^{n-m-2} \left[\left(\sqrt{\frac{n}{4}} + \sqrt{\frac{n-4}{4}} \right)^{m/d} - \left(-\sqrt{\frac{n}{4}} + \sqrt{\frac{n-4}{4}} \right)^{m/d} \right]^{2d}.
\end{aligned}$$

□

As a first consequence of Theorem 5 we can easily derive:

Corollary 3.

$$\begin{aligned}
T(K_n - C_3^1) &= n^{n-4}(n-3)^2,\ n \geq 3; \\
T(K_n - C_4^1) &= n^{n-5}(n-2)^2(n-4),\ n \geq 4; \\
T(K_n - C_5^1) &= n^{n-6}(n^2 - 5n + 5)^2,\ n \geq 5; \\
T(K_n - C_6^1) &= n^{n-7}(n-1)^2(n-3)^2(n-4),\ n \geq 6; \\
T(K_n - C_6^2) &= n^{n-6}(n-3)^4,\ n \geq 6.
\end{aligned}$$

The first four formulae of the above corollary already appear in [4] where they are given in generic forms and derived from *Kel'mans* and *Chelnokov*'s result (1) by direct computation.

The proof above illustrates our general tools. We now see how to apply them when looking at the complement of a more complicated circulant graph.

Theorem 6. *For* $n \geq m > 4$,

$$\begin{aligned}
T(K_n - C_m^{1,2}) = n^{n-m-2} &\left[\left(x_1 + \sqrt{x_1^2 - 1} \right)^m - \left(x_1 - \sqrt{x_1^2 - 1} \right)^m \right]^2 \cdot \\
&\left[\left(x_2 + \sqrt{x_2^2 - 1} \right)^m - \left(x_2 - \sqrt{x_2^2 - 1} \right)^m \right]^2,
\end{aligned}$$

where $x_1 = \sqrt{\frac{3}{8} + \frac{1}{8}\sqrt{25 - 4n}}$, $x_2 = \sqrt{\frac{3}{8} - \frac{1}{8}\sqrt{25 - 4n}}$.

Proof. We use a very similar technique to the proof of Theorem 5. In this proof let $\varepsilon_1 = e^{\frac{2\pi i}{m}}$, and x_1, x_2 be defined as above. Then

$$\begin{aligned} T(K_n - C_m^{1,2}) &= n^{n-m-1}\prod_{j=1}^{m-1}(n-4+\varepsilon_1^{-j}+\varepsilon_1^{-2j}+\varepsilon_1^{j}+\varepsilon_1^{2j}) \\ &= n^{n-m-1}\prod_{j=1}^{m-1}(n-4-12\cos^2(j\pi/m)+16\cos^4(j\pi/m)) \\ &= n^{n-m-1}16^{m-1}\prod_{j=1}^{m-1}(x_1^2-\cos^2(j\pi/m))((x_2^2-\cos^2(j\pi/m)) \\ &= n^{n-m-1}U_{m-1}^2(x_1)U_{m-1}^2(x_2). \end{aligned}$$

The closed formula in the theorem statement follows from (8) and then (5). □

As an application, Theorem 6 can directly implies the following formulae.

Corollary 4.

$$\begin{aligned} T(K_n - C_5^{1,2}) &= n^{n-6}(n-5)^4,\ n \geq 5; \\ T(K_n - C_6^{1,2}) &= n^{n-7}(n-6)^2(n-4)^3,\ n \geq 6; \\ T(K_n - C_7^{1,2}) &= n^{n-8}(n^3-14n^2+63n-91)^2,\ n \geq 7. \end{aligned}$$

We now examine the complement of a slightly more complicated circulant graph.

Theorem 7. *For $n \geq m > 8$, if m is odd, then*

$$T(K_n - C_m^{2,4}) = T(K_n - C_m^{1,2});$$

Otherwise, if m is even, then

$$\begin{aligned} T(K_n - C_m^{2,4}) &= n^{n-m-2}\left[\left(x_1+\sqrt{x_1^2-1}\right)^{m/2}-\left(x_1-\sqrt{x_1^2-1}\right)^{m/2}\right]^4 \cdot \\ &\left[\left(x_2+\sqrt{x_2^2-1}\right)^{m/2}-\left(x_2-\sqrt{x_2^2-1}\right)^{m/2}\right]^4, \end{aligned}$$

where x_1 and x_2 are as defined in Theorem 6.

Proof. If m is odd then $C_m^{2,4}$ is isomorphic to $C_m^{1,2}$ so the result of Theorem 6 applies. If m is even $C_m^{2,4}$ is the disjoint union of 2 circulant graphs $C_{m/2}^{1,2}$. The proof in this case is just to combine Corollary 1 and the proof of Theorem 6. When m is even then let $\varepsilon_2 = e^{\frac{4\pi i}{m}}$, we have

$$\begin{aligned}
T(K_n - C_m^{2,4}) &= T(K_n - C_{m/2}^{1,2} \bigcup C_{m/2}^{1,2}) \\
&= n^{n-m} \left(\prod_{j=1}^{\frac{m}{2}-1} (n - 4 + \varepsilon_2^{-j} + \varepsilon_2^{-2j} + \varepsilon_2^{j} + \varepsilon_2^{2j}) \right)^2 \\
&= n^{n-m} \left(\prod_{j=1}^{\frac{m}{2}-1} (n - 4 - 12\cos^2(2j\pi/m) + 16\cos^4(2j\pi/m)) \right)^2 \\
&= n^{n-m} \left(16^{\frac{m}{2}-1} \prod_{j=1}^{\frac{m}{2}-1} (x_1^2 - \cos^2(2j\pi/m))((x_2^2 - \cos^2(2j\pi/m)) \right)^2 \\
&= n^{n-m} U^4_{\frac{m}{2}-1}(x_1) U^4_{\frac{m}{2}-1}(x_2).
\end{aligned}$$

□

Corollary 5.

$$\begin{aligned}
T(K_n - C_9^{2,4}) &= n^{n-10}(n-6)^2(n^3 - 12n^2 + 45n - 51)^2, \ n \geq 9; \\
T(K_n - C_{10}^{2,4}) &= n^{n-10}(n-5)^8, \ n \geq 10; \\
T(K_n - C_{11}^{2,4}) &= n^{n-12}(n^5 - 22n^4 + 187n^3 - 759n^2 + 1441n - 979)^2, \ n \geq 11.
\end{aligned}$$

We now discuss the general technique for calculating $T(K_n - C_m^{s_1,s_2,\cdots,s_k})$ when $\gcd(s_1, s_2, \cdots, s_k, m) = 1$ (the case $\gcd(s_1, s_2, \cdots, s_k, m) \neq 1$ can then be dealt with similarly to the case "m is even" in the proof of Theorem 7). In the following paragraphs let $\varepsilon = e^{\frac{2\pi i}{m}}$, then from Corollary 1,

$$\begin{aligned}
&T(K_n - C_m^{s_1,s_2,\cdots,s_k}) \\
&= n^{n-m-1} \prod_{j=1}^{m-1} (n - 2k + \varepsilon^{-s_1 j} + \varepsilon^{-s_2 j} + \cdots + \varepsilon^{-s_k j} + \varepsilon^{s_1 j} + \varepsilon^{s_2 j} + \cdots + \varepsilon^{s_k j}) \\
&= n^{n-m-1} \prod_{j=1}^{m-1} (n - 2k + 2\cos(2s_1\pi/m) + 2\cos(2s_2\pi/m) + \cdots + 2\cos(2s_k\pi/m)).
\end{aligned}$$

It is easy to prove by induction that $\cos(kx)$ can be expressed as a polynomial in $\cos x$ of order k. Using this fact, for any integer s, $\cos(2sj\pi/m)$ can be written as a polynomial in $\cos^2(j\pi/m)$ of order s. So,

$$T(K_n - C_m^{s_1,s_2,\cdots,s_k}) = n^{n-m-1} \prod_{j=1}^{m-1} (n - 2k + g(\cos^2(j\pi/m))),$$

where $g(x)$ is a polynomial of order s_k. Thus,

$$T(K_n - C_m^{s_1,s_2,\cdots,s_k}) = n^{n-m-1} \prod_{j=1}^{m-1} f(\cos^2(\pi j/m)),$$

where $f(x)$ is a polynomial with degree s_k and the constant term is a linear function of n. Even more, by explicit calculation we can see that the coefficient of x^{s_k} in $f(x)$ is 4^{s_k}. We can therefore write

$$f(x) = (-4)^{s_k} \prod_{i=1}^{s_k} (x_i - \cos^2(\pi j/m)),$$

where $x_1, x_2, \cdots, x_k$ are zeros of $f(x)$. Then, combining formula (8) with last two equations we have

$$T(K_n - C_m^{s_1,s_2,\cdots,s_k}) = (-1)^{s_k} n^{n-m-1} \prod_{i=1}^{s_k} U_{m-1}^2(\sqrt{x_i}).$$

From this equation the exact formula for $T(K_n - C_m^{s_1,s_2,\cdots,s_k})$ can be derived from the properties of *Chebyshev* polynomials.

In the special cases $s_k \leq 4$, the polynomial $f(x)$ can be explicitly factored and an exact formula for the number of spanning trees in $K_n - C_m^{s_1,s_2,\cdots,s_k}$ as a function of n, m can therefore be derived. We have done this and the details are included in the full paper; in this extended abstract we omit these formulae.

From Corollary 2 and the properties of *Chebyshev* polynomials, we also can derive the following closed formulae for the numbers of spanning trees in complete graphs with circulant graphs added. As before we start by adding C_m^s.

Theorem 8. *For $n \geq m$, if $(m,\ s) = d$, then*

$$T(K_n + C_m^s) = n^{n-m-2}\left[\left(\sqrt{\frac{n+4}{4}} + \sqrt{\frac{n}{4}}\right)^{m/d} - \left(\sqrt{\frac{n+4}{4}} - \sqrt{\frac{n}{4}}\right)^{m/d}\right]^{2d}.$$

Proof. It's similar to the proof of Theorem 5. By Corollary 2, we have

$$\begin{aligned}
T(K_n + C_m^s) &= T(K_n + \bigcup_{u=1}^{d} C_{m/d}^1) \\
&= n^{n-m+d-2} \prod_{u=1}^{d} \prod_{j=1}^{\frac{m}{d}-1} (n + 2 - \varepsilon_1^{-j} - \varepsilon_1^{j}) \\
&= n^{n-m+d-2} \prod_{u=1}^{d} \prod_{j=1}^{\frac{m}{d}-1} (n + 2 - 2\cos(2dj\pi/m)) \\
&= n^{n-m+d-2} \prod_{u=1}^{d} \left[4^{\frac{m}{d}-1} \prod_{j=1}^{\frac{m}{d}-1} \left(\frac{n+4}{4} - \cos^2(dj\pi/m)\right)\right].
\end{aligned}$$

By using the formulae (8) and then (5),

$$\begin{aligned}
T(K_n + C_m^s) &= n^{n-m+d-2} \prod_{u=1}^{d} \left[U_{\frac{m}{d}-1}^2\left(\sqrt{\frac{n+4}{4}}\right)\right] \\
&= n^{n-m-2}\left[\left(\sqrt{\frac{n+4}{4}} + \sqrt{\frac{n}{4}}\right)^{m/d} - \left(\sqrt{\frac{n+4}{4}} - \sqrt{\frac{n}{4}}\right)^{m/d}\right]^{2d}.
\end{aligned}$$

□

This then immediately gives us, for example,

Corollary 6.

$$\begin{aligned}
T(K_n+C_2^1) &= n^{n-3}(n+4),\ n\geq 2;\\
T(K_n+C_3^1) &= n^{n-4}(n+3)^2,\ n\geq 3;\\
T(K_n+C_4^1) &= n^{n-5}(n+4)(n+2)^2,\ n\geq 4;\\
T(K_n+C_4^2) &= n^{n-4}(n+4)^2,\ n\geq 4;\\
T(K_n+C_5^1) &= n^{n-6}(n^2+5n+5)^2,\ n\geq 5.
\end{aligned}$$

We can also derive results for general $K_n + C_m^{s_1,s_2,\cdots,s_k}$ that are analogous to the ones previously derived for $K_n - C_m^{s_1,s_2,\cdots,s_k}$. Since the proofs are so similar, we omit them.

Theorem 9. *For $n \geq m$,*

$$\begin{aligned}
T(K_n+C_m^{1,2}) &= (-1)^m n^{n-m-2}\left[\left(x_1+\sqrt{x_1^2-1}\right)^m-\left(x_1-\sqrt{x_1^2-1}\right)^m\right]^2\cdot\\
&\left[\left(x_2+\sqrt{x_2^2-1}\right)^m-\left(x_2-\sqrt{x_2^2-1}\right)^m\right]^2,
\end{aligned}$$

where $x_1=\sqrt{\frac{3}{8}+\frac{1}{8}\sqrt{25+4n}}$, $x_2=\sqrt{\frac{3}{8}-\frac{1}{8}\sqrt{25+4n}}$.

Corollary 7.

$$\begin{aligned}
T(K_n+C_3^{1,2}) &= n^{n-4}(n+6)^2,\ n\geq 3;\\
T(K_n+C_4^{1,2}) &= n^{n-5}(n+4)(n+6)^2,\ n\geq 4;\\
T(K_n+C_5^{1,2}) &= n^{n-6}(n+5)^2,\ n\geq 5;\\
T(K_n+C_6^{1,2}) &= n^{n-7}(n+6)^2(n+4)^3,\ n\geq 6;\\
T(K_n+C_7^{1,2}) &= n^{n-8}(n^3+14n^2+63n+91)^2,\ n\geq 7.
\end{aligned}$$

Theorem 10. *For $n \geq m$, if m is odd, then*

$$T(K_n+C_m^{2,4}) = T(K_n+C_m^{1,2});$$

Otherwise m is even, then

$$\begin{aligned}
T(K_n+C_m^{2,4}) &= n^{n-m-2}\left[\left(x_1+\sqrt{x_1^2-1}\right)^{m/2}-\left(x_1-\sqrt{x_1^2-1}\right)^{m/2}\right]^4\cdot\\
&\left[\left(x_2+\sqrt{x_2^2-1}\right)^{m/2}-\left(x_2-\sqrt{x_2^2-1}\right)^{m/2}\right]^4,
\end{aligned}$$

where x_1 and x_2 are defined as in Theorem 9.

Corollary 8.

$$\begin{aligned}
T(K_n+C_6^{2,4}) &= n^{n-6}(n+6)^4,\ n\geq 6;\\
T(K_n+C_7^{2,4}) &= n^{n-8}(n^3+14n^2+63n+91)^2,\ n\geq 7;\\
T(K_n+C_8^{2,4}) &= n^{n-8}(n+4)^2(n+6)^4,\ n\geq 8;\\
T(K_n+C_9^{2,4}) &= n^{n-10}(n+6)^2(n^3+12n^2+45n+51)^2,\ n\geq 9.
\end{aligned}$$

4. Conclusion

In this paper we discussed how to use properties of *Chebyshev* polynomials to derive closed formulae for the number of spanning trees in $K_n \pm S$ where $S = C_m^{s_1,s_2,\cdots,s_k}$ is a circulant graph. Our key step was to factorize a polynomial of order s_k and then express the number of spanning trees in terms of *Chebyshev* polynomials evaluated at functions of the roots of the polynomial. In particular, when $s_k \leq 4$, we could explicitly factorize the polynomial and derive a"closed" form for the number of spanning trees.

One last thing that we should point it is that, in all the formulae we derived, we assumed that $s_1 < s_2 < \cdots < s_k$. This was just for the sake of convenience, though, and was not necessary for our proofs. The technique still works for reapeated s_i values, e.g., we could use it to evaluate $T(K_n + C_m^{1,1})$ $(m \leq n)$ where $C_m^{1,1}$ is the doubly-linked cycle.

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