## 4. Generalization to noncomplementary index sets

A more general inequality than (3.8) has the form

$$\Delta_{I,I'}\Delta_{K,K'} \leq \Delta_{J,J'}\Delta_{L,L'}. \tag{4.1}$$

That is, the pairs $(I, K)$, etc. need not be complements. However, any such inequality which holds for all totally nonnegative matrices can be deduced from Theorem 3.2 or equivalently from Corollary 3.4. We will give a combinatorial proof of this fact using the families of graphs introduced in Section 2.

Let $p$ and $p'$ be the cardinalities of $I \cup K$ and $I' \cup K'$, and let $q$ and $q'$ be the cardinalities of $I \cap K$ and $I' \cap K'$. Necessarily, $p + q = p' + q' = |I| + |K|$. Applying Observation 1.3, we may interpret the product of minors $\Delta_{I,I'}\Delta_{K,K'}$ of any totally nonnegative matrix to be the weighted sum of path families $\pi = (\pi_1, \ldots, \pi_{p+q})$ in a planar network which connect sources indexed by $I$ ($K$) to sinks indexed by $I'$ ($K'$) and in which all $S_I$ to $T_{I'}$ paths ($S_K$ to $T_{K'}$ paths) are vertex-disjoint. We will say that such a path family obeys *the* $(I, I', K, K')$ *crossing rule*. The following neccessary condition for eight sets to satisfy (4.1) for all totally nonnegative matrices was first stated in [3, Prop 2.2].

**Observation 4.1.** *Let $I, I', J, J', K, K', L, L'$ be subsets of $[m]$. Unless $I \cup K$ and $J \cup L$ are equal as multisets, and $I' \cup K'$ and $J' \cup L'$ are equal as multisets, the products $\Delta_{I,I'}\Delta_{K,K'}$ and $\Delta_{J,J'}\Delta_{L,L'}$ are incomparable as functions on totally nonnegative matrices.*

*Proof.* Suppose $i$ is an index which appears with greater multiplicity in $I \cup K$ than in $J \cup L$, and let $G$ be any planar network of order $m$ in which the unique edge leaving source $s_i$ has weight $c$. If $c$ is large enough, then the weighted path matrix of $G$ satisfies

$$\Delta_{I,I'}\Delta_{K,K'} > \Delta_{J,J'}\Delta_{L,L'}.$$

On the other hand, if $c$ is close enough to zero, we have the opposite strict inequality. Similarly, if $I' \cup K'$ and $J' \cup L'$ are not equal as multisets, then the products are again incomparable. □

Without loss of generality, we shall assume that the sets $I \cup K$ and $I' \cup K'$ are equal to $[p]$ and $[p']$, respectively. Otherwise we can delete appropriate matrix rows and columns to make this true. Necessary and sufficient conditions for eight sets to satisfy (4.1) are analogous to the inequalities (3.3) and (3.7). The appropriate choices of $I''$, $J''$, and $n$ are as follows. Let $n$ be the number $\frac{1}{2}(p-q+p'-q')$, let $\eta$ be the unique order preserving map

$$\eta : (I \smallsetminus K) \cup (K \smallsetminus I) \to [p-q],$$

and let $\eta'$ be the unique order *reversing* map

$$\eta' : (I' \smallsetminus K') \cup (K' \smallsetminus I') \to [p-q+1, 2n].$$

Define the subsets $I''$ and $J''$ of $[2n]$ by

$$I'' = \eta(I \smallsetminus K) \cup \eta'(K' \smallsetminus I'),$$
$$J'' = \eta(J \smallsetminus L) \cup \eta'(L' \smallsetminus J').$$

**Theorem 4.2.** *Let $I, I', J, J', K, K', L, L'$ be subsets of $[m]$, and define $p$, $p'$, $q$, $q'$, $n$, $\eta$, $\eta'$, $I''$, and $J''$, as above. Then the following statements are equivalent.*

(1) *In each totally nonnegative matrix of size at least $m \times m$, the minors $\Delta_{I,I'}$, $\Delta_{J,J'}$, $\Delta_{K,K'}$, $\Delta_{L,L'}$ satisfy*

$$\Delta_{I,I'}\Delta_{K,K'} \leq \Delta_{J,J'}\Delta_{L,L'}.$$

(2) *The multisets* $I \cup K$ *and* $J \cup L$ *are equal, the multisets* $I' \cup K'$ *and* $J' \cup L'$ *are equal, and the sets* $I''$, $J''$ *satisfy*

$$\max\{|B \cap I''|, |B \smallsetminus I''|\} \geq \max\{|B \cap J''|, |B \smallsetminus J''|\}$$

*for each even subinterval* $B$ *of* $[2n]$.

*Proof.* ($2 \Rightarrow 1$) Suppose (1) is false. Then there exists a planar network in which more path families obey the $(I, I'K, K')$ crossing rule than obey the $(J, J', L, L')$ crossing rule. This network contains a subnetwork $G$ which is a union of $(p+q)$ paths from $p$ sources to $p'$ sinks with the property that more path families $\pi = (\pi_1, \ldots, \pi_{p+q})$ which cover $G$ obey the $(I, I', K, K')$ crossing rule than obey the $(J, J', L, L')$ crossing rule.

Applying the procedure defining $\phi$ to $G$, we obtain a graph in which exactly $n$ connected components are paths whose endpoints belong to the $2n$-element set

$$S_{I \smallsetminus K} \cup S_{K \smallsetminus I} \cup T_{I' \smallsetminus K'} \cup T_{K' \smallsetminus I'}.$$

By the discussion following Corollary 2.2, these $n$ paths define a perfect matching of $S_{(I \smallsetminus K)} \cup T_{(K' \smallsetminus I')}$ with $S_{(K \smallsetminus I)} \cup T_{(I' \smallsetminus K')}$, which is not a perfect matching of $S_{(J \smallsetminus L)} \cup T_{(L' \smallsetminus J')}$ with $S_{(L \smallsetminus J)} \cup T_{(J' \smallsetminus L')}$. Let $H$ be the graph in $\mathcal{G}_3(2n)$ realizing this matching, in which vertex $i$ ($1 \leq i \leq p-q$) corresponds to the source in $S_{I \smallsetminus K} \cup S_{K \smallsetminus I}$ with the $i$th smallest index and vertex $j$ ($p-q < j \leq p-q+p'-q'$) corresponds to the sink in $T_{I' \smallsetminus K'} \cup T_{K' \smallsetminus I'}$ with the $j$th greatest index. Let $B = [b_1, b_2]$ be a minimal interval of $[2n]$ such that $(b_1, b_2)$ is an edge of $H$ and $b_1, b_2$ both belong to $J''$ or both belong to $\overline{J''}$. Then we have

$$\max\{|B \cap I''|, |B \smallsetminus I''|\} < \max\{|B \cap J''|, |B \smallsetminus J''|\}. \tag{4.2}$$

($1 \Rightarrow 2$) Let $B = [b_1, b_2]$ be a minimal subinterval of $[2n]$ which satisfies (4.2), and let $j_1$ and $j_2$ be the preimages of these numbers with respect to the maps $\eta$ and/or $\eta'$. Create a graph $H = (V, E)$ as follows.

(1) Place $2p + 2q$ vertices on a horizontal line.
(2) Define six sets of symbols

$$\begin{aligned} S &= \{s_i \,|\, i \in (I \smallsetminus K) \cup (K \smallsetminus I)\}, \\ U &= \{u_i \,|\, i \in (I \cap K)\}, \\ U' &= \{u'_i \,|\, i \in (I \cap K)\}, \\ T &= \{t_i \,|\, i \in (I' \smallsetminus K') \cup (K' \smallsetminus I')\}, \\ V &= \{v_i \,|\, i \in (I' \cap K')\}, \\ V' &= \{v'_i \,|\, i \in (I' \cap K')\}. \end{aligned}$$

(3) Label the leftmost $p+q$ vertices by the $p+q$ symbols $S \cup U \cup U'$, in order of nondecreasing indices. Label the rightmost $p+q$ vertices by the $p+q$ symbols $T \cup V \cup V'$, in order of nonincreasing indices.
(4) Connect each of the $q+q'$ pairs of the form $(u_i, u'_i)$ or $(v_i, v'_i)$ by $q+q'$ noncrossing arcs.
(5) Connect the $b_1$st (from the left) singleton to the $b_2$nd (from the left) singleton by an arc.
(6) Draw $p+q-1$ more arcs above the vertices to complete a noncrossing perfect matching of $S_{I \smallsetminus K} \cup T_{K' \smallsetminus I'}$ with $S_{K \smallsetminus I} \cup T_{I' \smallsetminus K'}$.

The arc drawn in step (5) prevents $H$ from inducing a perfect matching of $S_{J \smallsetminus L} \cup T_{L' \smallsetminus J'}$ with $S_{L \smallsetminus J} \cup T_{J' \smallsetminus L'}$. Now consider the planar network $G$ which is obtained from $H$ by creating $\rho(H)$ and then identifying all pairs of vertices of the form $(u_i, u'_i)$ or $(v_i, v'_i)$. There are no path families in $G$ which obey the $(J, J', L, L')$ crossing rule, but there are some which obey the $(I, I', K, K')$ crossing rule. This contradicts (1). $\square$

**Example 4.1.** Consider the two products of minors

$$\Delta_{\{1,2,3,6\},\{1,2,4,5\}}\Delta_{\{3,4\},\{2,5\}}, \quad \Delta_{\{1,3,6\},\{1,2,5\}}\Delta_{\{2,3,4\},\{2,4,5\}}.$$

We claim that these products are incomparable. To see this, let the map $\eta : \{1,2,4,6\} \to \{1,2,3,4\}$ preserve order, and let the map $\eta' : \{1,4\} \to \{5,6\}$ reverse order. Then define

$$I'' = \{\eta(1), \eta(2), \eta(6)\} \cup \emptyset = \{1,2,4\},$$
$$J'' = \{\eta(1), \eta(6)\} \cup \{\eta'(4)\} = \{1,4,5\}.$$

Since the sets $I'', J''$ satisfy

$$\max\{|[4,5] \cup I''|, |[4,5] \smallsetminus I''|\} < \max\{|[4,5] \cup J''|, |[4,5] \smallsetminus J''|\},$$
$$\max\{|[5,6] \cup I''|, |[5,6] \smallsetminus I''|\} > \max\{|[5,6] \cup J''|, |[5,6] \smallsetminus J''|\},$$

the products of minors are incomparable.

## 5. OPEN PROBLEMS

Note that the results of this paper reduce the problem of comparing minors in totally nonnegative matrices to the problem of counting *unweighted* path families in planar networks. This is curious, since Theorem 1.2 does not guarantee that for every totally nonnegative integer matrix $A$, there exists a planar network $G$ such that $A$ counts unweighted paths in $G$. This suggests the following open problem.

**Problem 5.1.** Characterize the totally nonnegative integer matrices $A = [a_{ij}]$ for which there exists a planar network $G$ such that each entry $a_{ij}$ counts *unweighted* paths in $G$ from source $s_i$ to sink $t_j$.

Other possibilities for extending the present work are the following.

**Problem 5.2.** Let $P(n)$ be the poset whose elements are set partitions of $[2n]$ into two blocks of size $n$, ordered by $I$-$\overline{I} \leq_{P(n)} J$-$\overline{J}$ whenever the inequality

$$\Delta_{I,I}\Delta_{\overline{I},\overline{I}} \leq \Delta_{J,J}\Delta_{\overline{J},\overline{J}}$$

holds for all totally nonnegative matrices. Find a simple description for $P(n)$.

**Problem 5.3.** Characterize the inequalities in products of $k$ minors which are satisfied by all totally nonnegative matrices, for $k > 2$.

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Department of Mathematics, University of Michigan, 2074 East Hall, 525 East University, Ann Arbor, Michigan, 48109-1109
*E-mail address*: mskan@umich.edu