# DISCRETE BOUNDARY-VALUE PROBLEMS (EXTENDED ABSTRACT)

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ABSTRACT. We consider partial linear recurrence equations in several dimensions over a domain obtained from the first orthant of the integer lattice by restricting it in one or more dimensions to an initial segment of the nonnegative integers. By means of the kernel method we obtain an explicit expression for the generating function of the solution in the case of a single restricted dimension, provided that the apex of the recurrence vanishes in all the remaining dimensions. Unlike the initial-value problem (where the domain is unrestricted in all dimensions) with apex of this type, this generating function depends rationally on the generating functions of the boundary conditions and of the right-hand side of the recurrence equation. As an example, we count lattice paths between two parallel hyperplanes with rational incline. By using the kernel method twice we also solve a discrete Dirichlet problem in two dimensions.

RÉSUMÉ. Nous considérons des relations de récurrence linéaires qui définissent des suites multi-dimensionnelles. Les points de $\mathbb{N}^d$ qui indexent ces suites appartiennent à une intersection de "tranches" de la forme $0 \leq x_i \leq m_i$, pour un certain nombre de directions contraintes $i$. Grâce à la méthode du noyau, nous obtenons une expression explicite de la série génératrice de ces suites dans le cas où une seule dimension est contrainte, à condition que l'apex de la récurrence soit nul dans toutes les autres directions. Contrairement au cas où aucune dimension n'est contrainte, cette série génératrice est une fonction rationnelle des séries génératrices données par les conditions initiales et du membre de droite de la relation de récurrence. Notre approche s'applique par exemple à l'énumération des chemins situés entre deux hyperplans de pente rationnelle. En utilisant deux fois la méthode du noyau, nous résolvons aussi un problème de Dirichlet en deux dimensions.

## 1. INTRODUCTION AND NOTATION

Let $D = D_1 \times D_2 \times \cdots \times D_d$ where each factor $D_i$ is either equal to $\mathbb{N}$ or to an initial segment $\{0, 1, \ldots, N_i\}$ of $\mathbb{N}$, for some $N_i \in \mathbb{N}$. In the latter case, we say that $D$ is *restricted in dimension $i$*. We study multivariate generating functions

$$F(\boldsymbol{x}) = F(x_1, \ldots, x_d) = \sum_{n_1 \in D_1, \ldots, n_d \in D_d} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d} = \sum_{\boldsymbol{n} \in D} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \tag{1}$$

whose coefficients satisfy a linear *recurrence relation* with constant coefficients

$$c_{\boldsymbol{h}_0} a_{\boldsymbol{n}+\boldsymbol{h}_0} + c_{\boldsymbol{h}_1} a_{\boldsymbol{n}+\boldsymbol{h}_1} + \cdots + c_{\boldsymbol{h}_k} a_{\boldsymbol{n}+\boldsymbol{h}_k} = b_{\boldsymbol{n}} \quad \text{for } \boldsymbol{n} \in R, \tag{2}$$

as well as *boundary conditions* of the form

$$a_{\boldsymbol{n}} = \varphi(\boldsymbol{n}), \qquad \text{for } \boldsymbol{n} \in D \setminus R, \tag{3}$$

where $c_{\boldsymbol{h}_i} \in \mathbb{C} \setminus \{0\}$, $\boldsymbol{h}_0 = \boldsymbol{0}$, $H = \{\boldsymbol{h}_0, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_k\} \subseteq \mathbb{Z}^d$ is the set of *shifts*, $R = \{\boldsymbol{n} \in \mathbb{N}^d;\ \boldsymbol{n} + H \subseteq D\}$ is the *range of validity* of (2), and $b : R \to \mathbb{C}$ as well as $\varphi : D \setminus R \to \mathbb{C}$ are known functions.

The *initial-value problem* where $D_i = \mathbb{N}$ for all $i$ has been treated in [2]. There it has been shown that under certain natural conditions, the solution of the initial-value problem

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exists and is unique. Here we consider the *boundary-value problem* where at least one $D_i$ is of the form $\{0, 1, \ldots, N_i\}$. In contrast to the initial-value problem, the boundary-value problem need not be solvable, or if it is, its solution need not be unique.

Similarly to [2], we define the *apex* $\boldsymbol{p}$ of (2) to be the componentwise maximum of all the shifts. Note that $\boldsymbol{p} \geq \boldsymbol{0}$ because we assume that $\boldsymbol{0} \in H$. Our main result is the following: when $D_1 = D_2 = \cdots = D_{d-1} = \mathbb{N}$, $D_d = \{0, 1, \ldots, N\}$, $p_1 = p_2 = \cdots = p_{d-1} = 0$, and $p_d \geq 0$, the generating function $F(\boldsymbol{x})$ of the solution is a *rational function*, provided that the generating functions of the right-hand side $b_{\boldsymbol{n}}$ and of the boundary values $\varphi(\boldsymbol{n})$ are rational. The degrees of the numerator and/or denominator of $F(\boldsymbol{x})$ may, however, depend on $N$. We provide an explicit formula which expresses $F(\boldsymbol{x})$ in terms of some *algebraic functions*. This is analogous to, say, Binet's formula for Fibonacci numbers which expresses an integer sequence in terms of powers of an algebraic number. Applications of our result include enumeration of lattice paths between parallel hyperplanes (cf. [6, 7]).

When more than one $D_i$ is bounded, we conjecture that the generating function $F(\boldsymbol{x})$ need not be holonomic as a function of both $\boldsymbol{x}$ and $N_i$. An example is furnished by the two-dimensional Gambler's Ruin problem (cf. [5, 4]).

The main tool that we use is the so-called *kernel method* which permits to solve certain systems of linear functional equations that seem to involve "too many unknowns". It works by restricting the equations to algebraic varieties on which some of the unknown terms vanish, thus providing the "missing equations" (see Section 3).

**Notation.** We use $\mathbb{N}$ to denote the set of nonnegative integers. We write $\boldsymbol{u} = (u_1, u_2, \ldots, u_d)$ for $d$-tuples of numbers or indeterminates, $\boldsymbol{0} = (0, 0, \ldots, 0)$, $\boldsymbol{u} \geq \boldsymbol{v}$ when $u_i \geq v_i$ for $1 \leq i \leq d$, and $\boldsymbol{u} > \boldsymbol{v}$ when $u_i > v_i$ for $1 \leq i \leq d$. The monomial $x_1^{u_1} \cdots x_d^{u_d}$ is denoted $\boldsymbol{x}^{\boldsymbol{u}}$.

## 2. From the recurrence relation to a functional equation

Instead of $F(\boldsymbol{x})$ as defined in (1) we consider

$$F_R(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in R} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}}.$$

The two generating functions $F(\boldsymbol{x})$ and $F_R(\boldsymbol{x})$ differ only by terms whose coefficients are given explicitly by the boundary conditions:

$$F(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in D} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} = \sum_{\boldsymbol{n} \in R} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} + \sum_{\boldsymbol{n} \in D \setminus R} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} = F_R(\boldsymbol{x}) + \sum_{\boldsymbol{n} \in D \setminus R} \varphi(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

Let us now transform our recurrence relation into a functional equation satisfied by the generating function $F_R(\boldsymbol{x})$. Multiplying (2) by $\boldsymbol{x}^{\boldsymbol{n}}$ and summing over all $\boldsymbol{n} \in R$ we obtain

$$\begin{aligned} \sum_{\boldsymbol{n} \in R} b_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} &= \sum_{\boldsymbol{n} \in R} \boldsymbol{x}^{\boldsymbol{n}} \sum_{\boldsymbol{h} \in H} c_{\boldsymbol{h}} a_{\boldsymbol{n}+\boldsymbol{h}} = \sum_{\boldsymbol{h} \in H} c_{\boldsymbol{h}} \boldsymbol{x}^{-\boldsymbol{h}} \sum_{\boldsymbol{n} \in R+\boldsymbol{h}} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \\ &= \sum_{\boldsymbol{h} \in H} c_{\boldsymbol{h}} \boldsymbol{x}^{-\boldsymbol{h}} \left[ F_R(\boldsymbol{x}) + P_{\boldsymbol{h}}(\boldsymbol{x}) - M_{\boldsymbol{h}}(\boldsymbol{x}) \right] \end{aligned} \tag{4}$$

where

$$P_{\boldsymbol{h}}(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in (R+\boldsymbol{h}) \setminus R} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} = \sum_{\boldsymbol{n} \in (R+\boldsymbol{h}) \setminus R} \varphi(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}} \quad \text{and} \quad M_{\boldsymbol{h}}(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in R \setminus (R+\boldsymbol{h})} a_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}}. \tag{5}$$

Here we used the obvious identity

$$\sum_{x \in A} f(x) = \sum_{x \in B} f(x) + \sum_{x \in A \setminus B} f(x) - \sum_{x \in B \setminus A} f(x)$$

valid for any sets $A$ and $B$ provided that all the indicated sums exist.

**Definition 1.** *The* apex *of (2) is the point* $\boldsymbol{p} = (p_1, p_2, \ldots, p_d) \in \mathbb{N}^d$ *defined by*

$$p_i = \max\{h_i : \boldsymbol{h} \in H\} \quad (i = 1, 2, \ldots, d).$$

*The* antiapex *of (2) is the point* $\boldsymbol{q} = (q_1, q_2, \ldots, q_d) \in \mathbb{N}^d$ *defined by*

$$q_i = \min\{h_i : \boldsymbol{h} \in H\} \quad (i = 1, 2, \ldots, d).$$

Multiplying (4) by $\boldsymbol{x}^{\boldsymbol{p}}$ where $\boldsymbol{p}$ is the apex of (2) we obtain

$$Q(\boldsymbol{x})F_R(\boldsymbol{x}) = U(\boldsymbol{x}) - K(\boldsymbol{x}) \tag{6}$$

where

$$Q(\boldsymbol{x}) = \sum_{\boldsymbol{h}\in H} c_{\boldsymbol{h}} \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{h}}, \tag{7}$$

$$K(\boldsymbol{x}) = \sum_{\boldsymbol{h}\in H} c_{\boldsymbol{h}} \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{h}} P_{\boldsymbol{h}}(\boldsymbol{x}) - \sum_{\boldsymbol{n}\in R} b_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{p}+\boldsymbol{n}}, \tag{8}$$

$$U(\boldsymbol{x}) = \sum_{\boldsymbol{h}\in H} c_{\boldsymbol{h}} \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{h}} M_{\boldsymbol{h}}(\boldsymbol{x}), \tag{9}$$

the series $P_{\boldsymbol{h}}$ and $M_{\boldsymbol{h}}$ being given by (5).

From the definition of the apex it follows that $Q(\boldsymbol{x})$ is a polynomial in $\boldsymbol{x}$ called the *characteristic polynomial* or the *kernel* of the recursion. Note that $Q(\boldsymbol{x})$ and $K(\boldsymbol{x})$ are given directly by the coefficients and the right-hand side of the recurrence relation, and by the boundary conditions.

The functional equation (6) seems to contain not one, but two unknown functions: $F_R$ and $U$. We shall show below on examples how to work with such apparently ambiguous functional equations. If $U(\boldsymbol{x})$ can be found explicitly then the generating function of the corresponding solution to (2), (3) is given by

$$F_R(\boldsymbol{x}) = \frac{U(\boldsymbol{x}) - K(\boldsymbol{x})}{Q(\boldsymbol{x})}. \tag{10}$$

## 3. A SINGLE RESTRICTED DIMENSION

**Theorem 1.** *Let* $D = \mathbb{N}^{d-1} \times \{0, 1, \ldots, N\}$ *and* $\boldsymbol{p} = (0, \ldots, 0, p)$. *If* $K(\boldsymbol{x})$ *is a rational function of* $\boldsymbol{x}$*, and the boundary-value problem* (2), (3) *is solvable, then it has a solution whose generating function* $F_R(\boldsymbol{x})$ *is a rational function of* $\boldsymbol{x}$*.*

*Proof:* In this case

$$\begin{aligned} R &= \{\boldsymbol{n} \in \mathbb{N}^d;\ \forall \boldsymbol{h} \in H : (\boldsymbol{n} + \boldsymbol{h} \geq \boldsymbol{0} \wedge n_d + h_d \leq N)\} \\ &= \{\boldsymbol{n} \in \mathbb{N}^d;\ \boldsymbol{n} + \boldsymbol{q} \geq \boldsymbol{0} \wedge n_d + p \leq N\} \end{aligned}$$

and

$$R + \boldsymbol{h} = \{\boldsymbol{n} \in \mathbb{N}^d;\ \boldsymbol{n} + \boldsymbol{q} \geq \boldsymbol{h} \wedge n_d + p \leq N + h_d\},$$

where $\boldsymbol{q}$ is the antiapex of $H$. In order to find $U(\boldsymbol{x})$ we wish to determine the sets $R\backslash(R+\boldsymbol{h})$ for all $\boldsymbol{h} \in H$. We distinguish three cases:

(1) $h_d = 0$: In this case $\boldsymbol{h} \leq \boldsymbol{0}$. If $\boldsymbol{n} \in R$ then $\boldsymbol{n} + \boldsymbol{q} \geq \boldsymbol{0} \geq \boldsymbol{h}$ and $n_d + p \leq N = N + h_d$, so $\boldsymbol{n} \in R + \boldsymbol{h}$. It follows that $R \subseteq R + \boldsymbol{h}$, hence $R \setminus (R + \boldsymbol{h}) = \emptyset$.

(2) $h_d > 0$: If $\boldsymbol{n} \in R$ then $n_d + p \le N < N + h_d$ and $\boldsymbol{n} + \boldsymbol{q} \ge \boldsymbol{0}$, so $n_i + q_i \ge 0 \ge h_i$ for $i \le d-1$. Therefore $\boldsymbol{n} \notin R + \boldsymbol{h}$ if and only if $n_d + q_d < h_d$ or, equivalently, $n_d \le h_d - q_d - 1$, hence

$$R \setminus (R + \boldsymbol{h}) = \{\boldsymbol{n} \in \mathbb{N}^d;\ -q_i \le n_i \ \text{ for } 1 \le i \le d-1,\ -q_d \le n_d \le h_d - q_d - 1\}.$$

(3) $h_d < 0$: In this case $\boldsymbol{h} \le \boldsymbol{0}$. If $\boldsymbol{n} \in R$ then $\boldsymbol{n} + \boldsymbol{q} \ge \boldsymbol{0} \ge \boldsymbol{h}$ and $n_d + p \le N$. Therefore $\boldsymbol{n} \notin R + \boldsymbol{h}$ if and only if $n_d + p < N + h_d$ or, equivalently, $n_d \ge N - p + h_d + 1$, hence

$$R \setminus (R + \boldsymbol{h}) = \{\boldsymbol{n} \in \mathbb{N}^d;\ -q_i \le n_i \ \text{ for } 1 \le i \le d-1,\ N - p + h_d + 1 \le n_d \le N - p\}.$$

From this and from (9) it follows that

$$\begin{aligned} U(\boldsymbol{x}) &= \sum_{\substack{\boldsymbol{h} \in H \\ h_d > 0}} c_{\boldsymbol{h}} \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{h}} \sum_{n_d = -q_d}^{h_d - q_d - 1} x_d^{n_d} f_{n_d}(x_1, \ldots, x_{d-1}) \\ &+ \sum_{\substack{\boldsymbol{h} \in H \\ h_d < 0}} c_{\boldsymbol{h}} \boldsymbol{x}^{\boldsymbol{p}-\boldsymbol{h}} \sum_{n_d = N-p+h_d+1}^{N-p} x_d^{n_d} f_{n_d}(x_1, \ldots, x_{d-1}) \end{aligned} \tag{11}$$

where

$$f_{n_d}(x_1, \ldots, x_{d-1}) = \sum_{(n_1, \ldots, n_{d-1}) \ge (q_1, \ldots, q_{d-1})} a_{n_1, \ldots, n_{d-1}, n_d} x_1^{n_1} \cdots x_{d-1}^{n_{d-1}},$$

for

$$-q_d \le n_d \le p - q_d - 1 \ \text{ or } \ N - p + q_d + 1 \le n_d \le N - p,$$

are $p - q_d$ unknown functions.

Let $\xi_1(x_1, \ldots, x_{d-1}), \ldots, \xi_r(x_1, \ldots, x_{d-1})$ be the distinct roots of $Q(\boldsymbol{x}) = 0$ considered as an equation in $x_d$, with respective multiplicities $m_1, \ldots, m_r$. Then $\sum_{k=1}^r m_k = \deg_{x_d} Q(\boldsymbol{x}) = p - q_d$. Notice that all the functions $F_R, Q, K, U$ are *polynomials* in $x_d$. Differentiating (6) $j$ times w.r.t. $x_d$ where $0 \le j < m_k$, then substituting $\xi_k$ for $x_d$ yields

$$\begin{aligned} &\sum_{\substack{\boldsymbol{h} \in H \\ h_d > 0}} c_{\boldsymbol{h}} x_1^{-h_1} \cdots x_{d-1}^{-h_{d-1}} \sum_{n_d = -q_d}^{h_d - q_d - 1} (n_d + p - h_d)^{\underline{j}} \xi_k^{n_d + p - h_d - j} f_{n_d} \\ &+ \sum_{\substack{\boldsymbol{h} \in H \\ h_d < 0}} c_{\boldsymbol{h}} x_1^{-h_1} \cdots x_{d-1}^{-h_{d-1}} \sum_{n_d = N-p+h_d+1}^{N-p} (n_d + p - h_d)^{\underline{j}} \xi_k^{n_d + p - h_d - j} f_{n_d} \\ &= \frac{\partial^j K(\boldsymbol{x})}{\partial x_d^j}\Big|_{x_d = \xi_k} \quad (1 \le k \le r,\ 0 \le j \le m_k - 1), \end{aligned} \tag{12}$$

a system of $p - q_d$ linear algebraic equations for the $p - q_d$ unknown $f_{n_d}$. Here $x^{\underline{j}}$ denotes the falling factorial power $x(x-1) \cdots (x-j+1)$.

The coefficients of the matrix and of the right-hand side of this system are rational functions of $x_i$ and $\xi_k$. Assume that the roots $\xi_k$ of $Q(\boldsymbol{x}) = 0$ considered as an equation in $x_d$ are all simple. From the way we constructed (12) it is clear that if (12) is solvable then it has a solution which is a rational function of $x_i$ and $\xi_k$, and which is *symmetric* in the variables $\xi_k$. It follows that this solution is a rational function of the elementary symmetric polynomials in $\xi_k$ (cf. [1, Ch. 14, Thm. 3.17]). These in turn are the coefficients of $Q(\boldsymbol{x})$ considered as a polynomial in $x_d$, hence they are polynomials in $x_1, \ldots, x_{d-1}$. Therefore the functions $f_{n_d}(x_1, \ldots, x_{d-1})$ are rational, and by (11), so are $U(\boldsymbol{x})$ and $F_R(\boldsymbol{x})$.

In the case when $Q(\boldsymbol{x})$ considered as a polynomial in $x_d$ has multiple roots we can perturb its coefficients slightly so that the roots become simple, then take the limit as the perturbation goes to 0 in the obtained rational solution.

**Remark 1.** When there are either no $\boldsymbol{h} \in H$ with $h_d > 0$, or no $\boldsymbol{h} \in H$ with $h_d < 0$, only a single term appears on the right side of (11) and we can determine $U(\boldsymbol{x})$ simply by substituting the roots $\xi_k$ for $x_d$ in (6), then using polynomial interpolation. Hence in these two special cases the problem (2), (3) is uniquely solvable.

### Example 1.: Lattice paths between two diagonal lines

Consider the problem of finding the number $a_{i,j}$ of two-dimensional lattice paths from the origin to the point $(i, j)$ which use the steps $(1, 0)$ and $(0, 1)$, and always stay within the diagonal strip $\{(x, y);\ x - b + 1 \le y \le x + c - 1\}$ where $b$ and $c$ are fixed positive integers. Applying the linear transformation $(x, y) \mapsto (x + y, -x + y)$ to the plane, we see that this number equals $u_{i+j,-i+j}$ where $u_{i,j}$ is the number of lattice paths from the origin to the point $(i, j)$ which use the steps $(1, -1)$ and $(1, 1)$, and always stay within the horizontal strip $\{(x, y);\ -b+1 \le y \le c-1\}$. In turn, the affine transformation $(x, y) \mapsto (x, y+b)$ shows that $u_{i,j} = v_{i,j+b}$ where $v_{i,j}$ is the number of lattice paths from $(0, b)$ to $(i, j)$ which use the steps $(1, -1)$ and $(1, 1)$ and always stay within the horizontal strip $\{(x, y);\ 1 \le y \le b + c - 1\}$. Clearly,

$$v_{i,j} = \begin{cases} v_{i-1,j-1} + v_{i-1,j+1}, & \text{when } i \ge 1 \text{ and } 1 \le j \le b + c - 1, \\ \delta_{(i,j),(0,b)}, & \text{when } i = 0 \text{ or } j = 0 \text{ or } j = b + c. \end{cases}$$

This is a boundary-value problem of the type (2), (3) with $D = \mathbb{N} \times \{0, 1, \ldots, b + c\}$, $R = \{1, 2, \ldots\} \times \{1, 2, \ldots, b+c-1\}$, $H = \{(-1, -1), (-1, 1)\}$ and $\boldsymbol{p} = (0, 1)$. The conditions of Theorem 1 are satisfied, so we expect a rational generating function. We find that

$$Q(x, y) = y - xy^2 - x$$

and the roots of $Q$ considered as a polynomial in $y$ are

$$\xi_{1,2}(x) = \frac{1 \pm \sqrt{1 - 4x^2}}{2x}.$$

From (8) and (9),

$$K(x, y) = -xy^b(y^2 + 1),$$

$$U(x, y) = -xy(f_1(x) + y^{b+c} f_{b+c-1}(x)),$$

and Equation (6) is

$$Q(x, y)F_R(x, y) = xy^b(y^2 + 1) - xy(f_1(x) + y^{b+c} f_{b+c-1}(x)).$$

After substituting $\xi_1(x)$ and $\xi_2(x)$ for $y$ in this equation, we obtain

$$\begin{aligned} f_1(x) &= \frac{\xi_1^c(x) - \xi_2^c(x)}{x(\xi_1^{b+c}(x) - \xi_2^{b+c}(x))}, \\ f_{b+c-1}(x) &= \frac{\xi_1^b(x) - \xi_2^b(x)}{x(\xi_1^{b+c}(x) - \xi_2^{b+c}(x))}, \end{aligned}$$

so

$$F(x, y) = F_R(x, y) + y^b = \frac{y^{b+1}}{y - xy^2 - x} - y\frac{\xi_1^c(x) - \xi_2^c(x) + y^{b+c}(\xi_1^b(x) - \xi_2^b(x))}{x(y - xy^2 - x)(\xi_1^{b+c}(x) - \xi_2^{b+c}(x))}.$$

Tracing back the transformations that we have made we find the generating function of the original problem as

$$\sum_{i,j\ge 0} a_{i,j}x^iy^j = \left(\sqrt{x/y}\right)^b F(\sqrt{xy},\sqrt{y/x}) = \frac{(2y)^c g(b) + (2x)^b g(c) - g(b+c)}{(x+y-1)g(b+c)} \tag{13}$$

where

$$g(u) = \left(1-\sqrt{1-4xy}\right)^u - \left(1+\sqrt{1-4xy}\right)^u.$$

Notice that for any fixed values of $b$ and $c$, (13) is a rational function of $x$ and $y$.

In a similar way we can treat the problems of lattice paths between any two parallel lines (or hyperplanes) of rational slope. The obtained generating functions are rational, in agreement with results of [6] and [7].

## 4. The discrete Dirichlet problem

When the domain of definition $D$ is restricted in all dimensions, the unknown sequence has only finitely many terms and so the boundary-value problem (2), (3) reduces to simple linear algebra. However, we wish to have an explicit formula for the generating function of the solution. Even though the conditions of Theorem 1 are not satisfied, the kernel method can sometimes provide the desired explicit solution. We illustrate this by providing an explicit solution of the two-dimensional Gambler's Ruin Problem [5, 4].

**Example 2.: Two-dimensional Gambler's Ruin**

In the one-dimensional Gambler's Ruin Problem two players start out with $i$ and $N-i$ dollars, respectively. At each step they toss a fair coin to decide who wins a dollar from the opponent. The game is over when one of them goes bankrupt. It is well known that the expected duration of the game is $i(N-i)$ (see [3], or almost any other textbook on probability).

In the two-dimensional variant [5] the players use two different currencies, say dollars and euros. They start out with ($i$ dollars, $j$ euros) and ($N-i$ dollars, $M-j$ euros), respectively. At each step they toss fair coins to decide the currency and the winner. The game is over when one of them runs out of either currency. What is the expected duration of the game?

Denote by game$(i,j)$ the game with the first player's initial assets equal to $(i,j)$. Assume that $1\le i\le N-1$ and $1\le j\le M-1$. Then after the first step, game$(i,j)$ turns into one of game$(i+1,j)$, game$(i-1,j)$, game$(i,j+1)$, or game$(i,j-1)$, each with probability 1/4. It follows that the expected duration $a_{i,j}$ of game$(i,j)$ satisfies the recurrence equation

$$a_{i,j} = \frac{a_{i+1,j}+a_{i-1,j}+a_{i,j+1}+a_{i,j-1}}{4} + 1 \qquad (1\le i\le N-1,\ 1\le j\le M-1) \tag{14}$$

and the boundary conditions

$$a_{0,j} = a_{N,j} = a_{i,0} = a_{i,M} = 0 \qquad (0\le i\le N,\ 0\le j\le M). \tag{15}$$

The unknown $a_{i,j}$, $1\le i\le N-1, 1\le j\le M-1$, can be obtained from (14), (15) by straightforward linear algebra. Instead of solving this linear system of $(N-1)(M-1)$ equations, Orr and Zeilberger [5] have shown how to obtain the values $a_{1,j} = a_{N-1,j}$ $(1\le j\le M-1)$ and $a_{i,1} = a_{i,M-1}$ $(1\le i\le N-1)$ from a system containing $O(N+M)$ equations only. Then all the remaining values $a_{i,j}$ can be computed from the recurrence (14). Here we provide explicit formulas for $a_{1,j}$ and $a_{i,1}$ using the kernel method twice in a row, at two different levels. Thus we avoid the need to solve linear systems altogether.

Writing (14) as $a_{i,j} - (a_{i+1,j}+a_{i-1,j}+a_{i,j+1}+a_{i,j-1})/4 = 1$, we have a boundary-value problem of the form (2), (3) with $D = \{0,1,\ldots,N\}\times\{0,1,\ldots,M\}$, $H =$

$\{(0,0),(1,0),(-1,0),(0,1),(0,-1)\}$, $R=\{1,2,\ldots,N-1\}\times\{1,2,\ldots,M-1\}$, $b_{i,j}=1$, $\boldsymbol{p}=(1,1)$,

$$Q(x,y)=xy-\frac{1}{4}(y+x^2y+x+xy^2),$$
$$K(x,y)=-\sum_{(i,j)\in R}b_{i,j}x^{i+1}y^{j+1}=-xy\frac{x^N-x}{x-1}\cdot\frac{y^M-y}{y-1},$$
$$U(x,y)=-\frac{1}{4}xy\left(f_1(x)+f_2(y)+y^Mg_1(x)+x^Ng_2(y)\right)$$

where

$$f_1(x)=\sum_{i=1}^{N-1}a_{i,1}x^i,\qquad f_2(y)=\sum_{j=1}^{M-1}a_{1,j}y^j,$$
$$g_1(x)=\sum_{i=1}^{N-1}a_{i,M-1}x^i,\qquad g_2(y)=\sum_{j=1}^{M-1}a_{N-1,j}y^j.$$

As the game is symmetric with respect to the players, $a_{i,1}=a_{i,M-1}$ and $a_{1,j}=a_{N-1,j}$, so $f_1(x)=g_1(x)$ and $f_2(y)=g_2(y)$. Also, because of the zero boundary conditions, $F_R(x,y)=F(x,y)$. Thus the functional equation (6) has the form

$$Q(x,y)F(x,y)=xy\left(\frac{x^N-x}{x-1}\cdot\frac{y^M-y}{y-1}-\frac{1}{4}\left(f_1(x)(1+y^M)+f_2(y)(1+x^N)\right)\right)\tag{16}$$

where $F(x,y)$, $f_1(x)$, and $f_2(y)$ are unknown polynomials of degrees in $x$ and $y$ not exceeding $N-1$ and $M-1$, respectively.

To apply the kernel method to equation (16), let

$$\xi(x)=-\frac{1}{2x}\left(x^2-4x+1+(x-1)\sqrt{x^2-6x+1}\right)\tag{17}$$

be the analytic root of $Q(x,y)=0$ considered as an equation in $y$:

$$\xi(x)=-x-4\,x^2-16\,x^3-68\,x^4-304\,x^5-1412\,x^6-6752\,x^7-\cdots$$

One can show that $\xi(\xi(x))=x$. Substituting $\xi(x)$ for $y$ in (16) we have

$$f_1(x)(1+\xi^M(x))+f_2(\xi(x))(1+x^N)=4\frac{x^N-x}{x-1}\cdot\frac{\xi^M(x)-\xi(x)}{\xi(x)-1},\tag{18}$$

reducing the number of unknowns from three to two. To reduce it further, we apply the kernel method again, this time to equation (18). Write

$$\omega_N=e^{\frac{\pi i}{N}},\qquad \omega_{k,N}=\omega_N^{2k+1}\quad(k=0,1,\ldots,N-1).\tag{19}$$

Note that $\omega_{k,N}$ are the $N$-th roots of $-1$.

**Lemma 1.** *Let $\xi(x)$ and $\omega_{k,N}$ be as in (17) and (19), respectively. Then:*

(i) $\omega_{k,N}\neq 1$,
(ii) $\xi(\omega_{k,N})\neq 1$,
(iii) $\xi(\omega_{k,N})^M\neq -1$.

*Proof:* (i) is obvious as $\omega_{k,N}^N=-1$. The assertion $\xi(x)=1$ is equivalent to $Q(x,1)=0$. But $Q(x,1)=-(x-1)^2/4$, so $\xi(x)=1$ if and only if $x=1$. Thus (ii) follows from (i). To prove (iii), write $\varphi_{k,N}=(2k+1)\pi/(2N)$. Then $Q(\omega_{k,N},y)=-\omega_{k,N}\left(y^2-2y\left(2\sin^2\varphi_{k,N}+1\right)+1\right)/4$, whence

$$\xi\left(\omega_{k,N}\right)=\left(\sin\varphi_{k,N}-\sqrt{\sin^2\varphi_{k,N}+1}\right)^2.$$

As a positive real number, $\xi\left(\omega_{k,N}\right)$ is not an $M^{\text{th}}$ root of $-1$. □

Substitution of $\omega_{k,N}$ for $x$ in (18), justified by Lemma 1, yields

$$f_1(\omega_{k,N}) = \alpha_{k,N,M} \qquad (k = 0, 1, \ldots, N-1),$$

where

$$\alpha_{k,N,M} = \frac{4\,i\cot\varphi_{k,N}}{1+\xi^M(\omega_{k,N})}\cdot\frac{\xi^M(\omega_{k,N})-\xi(\omega_{k,N})}{\xi(\omega_{k,N})-1} \qquad (i^2=-1). \tag{20}$$

These $N$ values uniquely determine the unknown polynomial $f_1(x)$. To find its coefficients explicitly we use the inversion formula

$$b_k = \sum_{i=0}^{N-1} a_i\,\omega_{k,N}^i \quad (0 \le k \le N-1) \quad \Longleftrightarrow \quad a_i = \frac{1}{N}\sum_{k=0}^{N-1} b_k\,\omega_{k,N}^{-i} \quad (0 \le i \le N-1)$$

which is verifiable by straightforward computation. Writing $f_1(x) = \sum_{i=0}^{N-1} a_i x^i$ and $b_k = f_1(\omega_{k,N})$, we obtain an explicit expression for the unknown coefficients of $f_1(x)$

$$a_{i,1} = a_i = \frac{1}{N}\sum_{k=0}^{N-1}\frac{\alpha_{k,N,M}}{\omega_{k,N}^i}. \tag{21}$$

In the same way we obtain an explicit expression for the unknown coefficients of $f_2(y)$,

$$a_{1,j} = \frac{1}{M}\sum_{k=0}^{M-1}\frac{\alpha_{k,M,N}}{\omega_{k,M}^i}. \tag{22}$$

$$f_1(x) = \frac{1}{N}\sum_{j=0}^{N-1} x^j \sum_{k=0}^{N-1}\frac{\alpha_{k,N,M}}{\omega_{k,N}^j} \tag{23}$$

where $\alpha_{k,N,M}$ and $\omega_{k,N}$ are given in (20) and (19), respectively.

By interchanging the order of summations in (23), or by using Lagrange Interpolation Formula on the data from (20), we can express $f_1(x)$ with a single summation sign:

$$f_1(x) = \frac{x^N+1}{N}\sum_{k=0}^{N-1}\frac{\alpha_{k,N,M}}{1-x\omega_{k,N}^{-1}}.$$

In the same way we obtain the other unknown function

$$f_2(y) \;=\; \frac{1}{M}\sum_{j=0}^{M-1} y^j \sum_{k=0}^{M-1}\frac{\alpha_{k,M,N}}{\omega_{k,M}^j} \;=\; \frac{y^M+1}{M}\sum_{k=0}^{M-1}\frac{\alpha_{k,M,N}}{1-y\omega_{k,M}^{-1}}.$$

Finally we have the following explicit expression for the entire generating function $F(x,y) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}x^i y^j$:

$$F(x,y) = \frac{xy\left(4\,\frac{x^N-x}{x-1}\,\frac{y^M-y}{y-1} - (x^N+1)(y^M+1)\left(\frac{1}{N}\sum_{k=0}^{N-1}\frac{\alpha_{k,N,M}}{1-x\omega_{k,N}^{-1}} + \frac{1}{M}\sum_{k=0}^{M-1}\frac{\alpha_{k,M,N}}{1-y\omega_{k,M}^{-1}}\right)\right)}{4\,xy-(x+y)(1+xy)}.$$

In closing we note that the values $a_{i,j}$ can be given explicitly as double trigonometric sums either using the Discrete Fourier Transform, or by direct diagonalization of the linear system (14) (cf. [4]).

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