# TRIBASIC INTEGRALS AND IDENTITIES OF ROGERS-RAMANUJAN TYPE

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ABSTRACT. Using complex analysis, integrals involving three independent $q$'s are evaluated as infinite products. This leads to identities of Rogers-Ramanujan type.

RÉSUMÉ. En utilisant trois $q$ independant, nous evaluons des integraux en analyse complexe et trouver des produits infinis. Ceci nous donnons des generalisations des identites de Rogers-Ramanujan.

## 1. INTRODUCTION

The purpose of this extended abstract (no proofs are given) is to show the extensive relationship between integrals and identities of Rogers-Ramanujan type. We concentrate on integral evaluations involving infinite products with three independent $q$'s.

In [4, 6], it was shown that the Rogers-Ramanujan identities of modulus 5 follow from evaluating the bibasic integral

$$\frac{(qt^2;q)_\infty}{2\pi}\int_0^\pi \frac{(q^5,e^{2i\theta},e^{-2i\theta};q^5)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}d\theta. \tag{1}$$

We generalize the above integral by replacing the three bases of the infinite products $q^5$, $q$ and $q$ by independent bases $s$, $p$ and $q$. The resulting integral can be evaluated as an infinite product for special values of $t$. By specializing $s$, $p$, and $q$ the product sides of identities of Rogers-Ramanujan type appear. The special values of $t$ for which such identities exist can be found by considering the singularities of the integrals as functions of $t$.

## 2. INTEGRALS

We use a variant of Schwarz's theorem [1] to evaluate limits of integrals.

**Theorem 1.** *Let $f(\theta,z)$ be continuous in $\theta$ for $\theta\in[0,2\pi]$, and for all $z$ so that $r\ge|z|\ge r-\epsilon$ for some positive $\epsilon$. Assume further that $f(\theta,z)$ converges to $f(\theta,re^{i\phi})$ as $z\to re^{i\phi}$ uniformly in $\theta$, for $\theta\in[0,2\pi]$. Then*

$$\lim_{z\to re^{i\phi}}\int_0^{2\pi}\frac{(r^2-|z|^2)\,f(\theta,z)d\theta}{2\pi|re^{i\theta}-z|^2}=f(\phi,re^{i\phi}).$$

We next apply Theorem 1 to find limiting values of the tribasic integrals

$$G_1(t,p,q,s)=\frac{(-t;q)_\infty}{2\pi}\int_0^\pi\frac{(s,e^{2i\theta},e^{-2i\theta};s)_\infty}{(te^{2i\theta},te^{-2i\theta};p)_\infty}\,d\theta,\quad |t|<1,$$

and

$$G_2(t,p,q,s)=\frac{(pt^2;q)_\infty}{2\pi}\int_0^\pi\frac{(s,e^{2i\theta},e^{-2i\theta};s)_\infty}{(te^{i\theta},te^{-i\theta};p)_\infty}\,d\theta,\quad |t|<1.$$

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Note that $G_2(t,q,q,q^5)$ is the integral (1) of the introduction giving the Rogers-Ramanujan identities modulo 5.

**Corollary 2.** *We have an analytic continuation of $G_2(t,p,q,s)$ to $p^{1/2}<|t|<p^{-1/2}$ such that*

$$\begin{aligned}\lim_{t\to -1} G_1(t,p,q,s) &= \frac{(q;q)_\infty}{(p,p;p)_\infty}(s,-s,-s;s)_\infty,\\ \lim_{t\to p^{-1/2}} G_2(t,p,q,s) &= \frac{(s,p,s/p;s)_\infty(q;q)_\infty}{(p,p;p)_\infty}.\end{aligned}$$

In particular for the Rogers-Ramanujan identities modulo 5,

$$G_2(q^{-1/2},q,q,q^5)=\frac{1}{(q^2,q^3;q^5)_\infty}.$$

One can also extend the evaluation for $G_2$ using further analytic continuations. Let $F(z)=(s,z^2,1/z^2;s)_\infty$.

**Theorem 3.** *For $k=1,2,\ldots,$ $G_2(t,p,p,s)$ can be analytically continued to*

$$p^{1-k/2}<|t|<p^{-k/2},\ \textit{via}$$

$$G_2(t,p,p,s)=\frac{(pt^2;p)_\infty}{4\pi}\int_0^{2\pi}\frac{F(e^{i\theta}p^{-k/2})}{(te^{i\theta}p^{-k/2},tp^{k/2}e^{-i\theta};p)_\infty}\,d\theta$$

$$+\frac{(s;s)_\infty}{2(1-t^2)(p;p)_\infty}\sum_{j=0}^{k-1}\frac{(t^2;p)_j(t^2p^{2j},t^{-2}p^{-2j};s)_\infty}{(1/p;1/p)_j}.$$

*Furthermore*

$$\lim_{t\to p^{-k/2}} G_2(t,p,p,s)=\frac{1}{2(1-p^{-k})(p;p)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\ j\end{bmatrix}_p(s,p^{2j-k},p^{k-2j};s)_\infty p^{j(j-k)}.$$

## 3. Sums

Using techniques from orthogonal polynomials, one can give power series representations in $t$ for several special choices of $p$ in the integrals

$$S_{p,q}(t)=G_2(t,q,q,p)=\frac{(qt^2;q)_\infty(p;p)_\infty}{2\pi}\int_0^{\pi}\frac{(e^{2i\theta},e^{-2i\theta};p)_\infty}{(te^{i\theta},te^{-i\theta};q)_\infty}d\theta,$$

$$H_{p,q}(t)=G_2(t,p,p^2,q^2)=\frac{(q^2;q^2)_\infty(pt^2;p^2)_\infty}{2\pi}\int_0^{\pi}\frac{(e^{2i\theta},e^{-2i\theta};q^2)_\infty}{(te^{i\theta},te^{-i\theta};p)_\infty}d\theta,$$

$$J_{p,q}(t)=G_1(t,p^2,p,q)=\frac{(q;q)_\infty(-t;p)_\infty}{2\pi}\int_0^{\pi}\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(te^{2i\theta},te^{-2i\theta};p^2)_\infty}d\theta.$$

For example

$$
\begin{aligned}
S_{q^5,q}(t) &= \sum_{n=0}^{\infty} \frac{q^{n^2+2n}t^{2n}}{(q;q)_n}, \\
S_{q^5,q^2}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_{2n}} q^{n(3n+2)}(-t^2)^n, \\
S_{q^7,q^2}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_{2n}} q^{2n(n+2)}t^{2n}, \\
S_{-q^3,q}(t) &= \sum_{n=0}^{\infty} \frac{(-1;q^2)_n}{(q;q)_{2n}} q^{n(n+2)}t^{2n}. \\
S_{\omega q^3,q}(t) &= 1+(1-\omega)\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}}{(q;q)_{2n}(q;q)_{n-1}} q^{n(n+2)}t^{2n}.
\end{aligned}
$$

$$
\begin{aligned}
H_{q^2,q}(t) &= \sum_{n=0}^{\infty} \frac{q^{2n^2}(-t^2)^n}{(q^4;q^4)_n} = (q^2t^2;q^4)_\infty, \\
H_{iq,q}(t) &= \sum_{n=0}^{\infty} \frac{(-1;q^4)_n}{(iq;iq)_{2n}}(-qt)^{2n}, \\
H_{q,q^2}(t) &= \sum_{n=0}^{\infty} \frac{(q^2;q^4)_n}{(q;q)_{2n}}(q^2t^2)^n = \frac{(-q^3t^2;q^2)_\infty}{(q^2t;q^2)_\infty}, \\
H_{q,q^3}(t) &= \sum_{n=0}^{\infty} \frac{(-q;q)_{2n}}{(q^2;q^2)_n}(qt)^{2n}, \\
H_{q^2,q^3}(t) &= \sum_{n=0}^{\infty} \frac{(q^4t^2)^n}{(q^4;q^4)_n} = \frac{1}{(q^4t^2;q^4)_\infty},
\end{aligned}
$$

$$
\begin{aligned}
J_{q^2,q}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^4;q^4)_n} q^{n^2}(-t)^n, \\
J_{-q,q}(t) &= \sum_{n=0}^{\infty} \frac{(-1;q^2)_n}{(q^2;q^2)_n}(-qt)^n = \frac{(qt;q^2)_\infty}{(-qt;q^2)_\infty}, \\
J_{q,q^2}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_n}(qt)^n = \frac{(q^2t;q^2)_\infty}{(qt;q^2)_\infty}, \\
J_{q,q^3}(t) &= \sum_{n=0}^{\infty} \frac{(q;q)_{2n}}{(q;q)_n(q^2;q^2)_n}(qt)^n, \\
J_{q^2,q^3}(t) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^4;q^4)_n}(q^2t)^n.
\end{aligned}
$$

Note that several of these series converge for all $t$, thus are analytic continuations of the integrals, allowing one to specialize $t$, and apply Corollary 2 and Theorem 3. These are Rogers-Ramanujan identities. For example, specializing $S_{\omega q^3,q}(t)$, $\omega = e^{2\pi i/3}$, yields

**Theorem 4.** *We have*

$$1 + (1-\omega)\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}}{(q;q)_{2n}(q;q)_{n-1}} q^{n(n+1)} = \frac{(\omega q^3, \omega q^2, q; \omega q^3)_\infty}{(q;q)_\infty}$$
$$1 + (1-\omega)\sum_{n=1}^{\infty} \frac{(q^3;q^3)_{n-1}}{(q;q)_{2n}(q;q)_{n-1}} q^{n^2} = \frac{(\omega q^3, q^2, \omega q; \omega q^3)_\infty}{(q;q)_\infty}.$$

## 4. Integer partition interpretations

Several of the identities are equivalent to integer partition statements, here are three examples.

**Corollary 5.** *Let $A(n)$ be the number of integer partitions of $n$ into parts congruent to 2,4, 10 or 12 mod 14 and distinct parts congruent to 1,5,7,9, or 13 mod 14. Let $B(n)$ be the number of integer partitions of $n$*
*(1) whose odd parts are consecutive (starting with 1) and have multiplicity one or two,*
*(2) whose largest even part is at most two more than twice the largest odd part.*
*Then $A(n) = B(n)$.*

**Corollary 6.** *Let $A(n)$ be the number of integer partitions of $n$ into parts congruent to 2,6,8, or 12 mod 14 and distinct parts congruent to 3,7, or 11 mod 14. Let $B(n)$ be the number of integer partitions of $n$*
*(1) whose odd parts are consecutive (starting with 1) and have multiplicity two or three,*
*(2) whose largest even part is at most two more than twice the largest odd part.*
*Then $A(n) = B(n)$.*

**Corollary 7.** *Let $A(n)$ be the number of integer partitions of $n$ into parts congruent to 4 or 6 mod 10 and distinct parts congruent to 3,5, or 7 mod 10. Let $B(n)$ be the number of integer partitions of $n$*
*(1) whose odd parts are consecutive (starting with 1) and have multiplicity three or four,*
*(2) whose largest even part is at most two more than twice the largest odd part.*
*Then $A(n) = B(n)$.*

## 5. $m$-versions

The Rogers-Ramanujan identities have the natural generalization [4]

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = \frac{a_m(q)}{(q,q^4;q)_\infty} + \frac{b_m(q)}{(q^2,q^3;q)_\infty}, \tag{2}$$

where $a_m(q)$ and $b_m(q)$ are Laurent polynomials in $q$ which are explicitly known. We refer to (2) as an "$m$-version" of the Rogers-Ramanujan identities.

According to Theorem 3

$$\lim_{t\to q^{-m/2}} S_{p,q}(t) = \frac{1}{2(1-q^{-m})} \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q \frac{(p, q^{2j-m}, q^{m-2j}; p)_\infty}{(q;q)_\infty} q^{j(j-m)}, \tag{3}$$

for $m = 1, 2, \ldots$. Equation (3) generalizes (2) and gives an explicit form for the generalizations of the polynomials $a_m(q)$ and $b_m(q)$. This alternating form is a special case of the hook difference polynomials in [3]. However explicit positive forms may be found using the recurrence relations for the polynomials. In the mod 5 case, this recurrence is three-term and related to orthogonal polynomials. But in other cases higher order recurrences do occur.

For example

$$\begin{aligned} S_{q^7,q^2}(q^{-m}) &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_{2n}}\, q^{2n^2+4n-2mn} \\ &= c_1(m,7,2)\frac{(q^7,q^1,q^6;q^7)_\infty}{(q^2,q^2)_\infty} + c_2(m,7,2)\frac{(q^7,q^2,q^5;q^7)_\infty}{(q^2,q^2)_\infty} \\ &+ c_3(m,7,2)\frac{(q^7,q^3,q^4;q^7)_\infty}{(q^2,q^2)_\infty}. \end{aligned}$$

where $g_m = c_i(m+2,7,2)$ satisfies the four term recurrence relation

$$g_{m+2} + q^{-1}g_{m+1} - (1+q^{-2-2m})g_m - q^{-1}g_{m-1} = 0.$$

The explicit positive forms are $(s = q^{-1})$

$$\begin{aligned} c_1(n+2,7,2) &= (-1)^{n-1} \sum_{2m+2j+k+1=n} \begin{bmatrix} m+j \\ j \end{bmatrix}_{s^4} \begin{bmatrix} m+k \\ k \end{bmatrix}_{s^2} s^{2m(m+1)+k} \\ c_2(n+2,7,2) &= (-1)^{n-1} \sum_{2m+2j+k+1=n} \begin{bmatrix} m+j \\ j \end{bmatrix}_{s^4} \begin{bmatrix} m+k-1 \\ k \end{bmatrix}_{s^2} s^{2m(m+1)+k}, \\ c_3(n+2,7,2) &= (-1)^{n} \sum_{2m+2j+k=n} \begin{bmatrix} m+j \\ j \end{bmatrix}_{s^4} \begin{bmatrix} m+k-1 \\ k \end{bmatrix}_{s^2} s^{2m^2+k}. \end{aligned}$$

One may also find these explicit positive forms for $n < 0$.

Of the identities on Slater's list [5], 39 involve 3-term recurrences and 59 involve 4-term recurrences.

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