

THE NUMBER OF REPRESENTATIONS OF A NUMBER BY VARIOUS FORMS

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ABSTRACT. We find formulae for the number of representations of the integer n as, for example, the sum of two triangles and two squares, or of four triangles, in terms of divisor functions. Indeed, we find sixteen formulae of this type, a few of which are known, the remainder apparently new.

RÉSUMÉ. Nous trouvons des formules pour le nombre des représentations d'un entier n comme, par exemple, somme de deux nombres triangulaires et de deux carrés, ou de quatre nombres triangulaires, en termes des fonctions des diviseurs de n . En effet, nous trouvons seize formules de ce type, dont quelques unes sont connues, les autres apparemment nouvelles.

1. INTRODUCTION

There are several classical results which give the number of representations of a number by a quadratic form in terms of a divisor function. The object of this note is to consider four such results and from them to derive many more of the same sort.

With $d_{r,m}(n)$ denoting the number of divisors d of n with $d \equiv r \pmod{m}$ and $\sigma(n)$ the sum of the divisors of n , the results we consider are the following. Proofs of all four can be found in [2].

Theorem 1. (Jacobi, 1828). *The number of representations of $n \geq 1$ as the sum of two squares is*

$$(J1) \quad r\{\square + \square\}(n) = 4(d_{1,4}(n) - d_{3,4}(n)).$$

Theorem 2. (Dirichlet, 1840). *The number of representations of $n \geq 1$ as the sum of a square and twice a square is*

$$(D) \quad r\{\square + 2\square\}(n) = 2(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)).$$

Theorem 3. (L. Lorenz, 1871). *The number of representations of $n \geq 1$ as the sum of a square and three times a square is*

$$(L) \quad r\{\square + 3\square\}(n) = 2(d_{1,3}(n) - d_{2,3}(n)) + 4(d_{4,12}(n) - d_{8,12}(n)).$$

Theorem 4. (Jacobi, 1829). *The number of representations of $n \geq 1$ as the sum of four squares is*

$$(J2) \quad r\{\square + \square + \square + \square\}(n) = 8 \sum_{d|n, 4 \nmid d} d = 8 \left(\sigma(n) - 4\sigma\left(\frac{n}{4}\right) \right).$$

We shall prove the following sixteen results of the same sort.

- (1.1) $r\{\Delta + \Delta\}(n) = d_{1,4}(4n+1) - d_{3,4}(4n+1),$
(1.2) $r\{\square + 2\Delta\}(n) = d_{1,4}(4n+1) - d_{3,4}(4n+1),$
(1.3) $r\{2\square + \Delta\}(n) = d_{1,4}(8n+1) - d_{3,4}(8n+1),$
(1.4) $r\{\Delta + 4\Delta\}(n) = \frac{1}{2}(d_{1,4}(8n+5) - d_{3,4}(8n+5)),$
(1.5) $r\{\Delta + 2\Delta\}(n) = \frac{1}{2}(d_{1,8}(8n+3) + d_{3,8}(8n+3) - d_{5,8}(8n+3) - d_{7,8}(8n+3)),$
(1.6) $r\{\square + \Delta\}(n) = d_{1,8}(8n+1) + d_{3,8}(8n+1) - d_{5,8}(8n+1) - d_{7,8}(8n+1),$
(1.7) $r\{\square + 4\Delta\}(n) = d_{1,8}(2n+1) + d_{3,8}(2n+1) - d_{5,8}(2n+1) - d_{7,8}(2n+1),$
(1.8) $r\{\Delta + 3\Delta\}(n) = d_{1,3}(2n+1) - d_{2,3}(2n+1),$
(1.9) $r\{3\square + 2\Delta\}(n) = d_{1,3}(4n+1) - d_{2,3}(4n+1),$
(1.10) $r\{\square + 6\Delta\}(n) = d_{1,3}(4n+3) - d_{2,3}(4n+3),$
(1.11) $r\{6\square + \Delta\}(n) = d_{1,3}(8n+1) - d_{2,3}(8n+1),$
(1.12) $r\{\Delta + 12\Delta\}(n) = \frac{1}{2}(d_{1,3}(8n+13) - d_{2,3}(8n+13)),$
(1.13) $r\{2\square + 3\Delta\}(n) = d_{1,3}(8n+3) - d_{2,3}(8n+3),$
(1.14) $r\{3\Delta + 4\Delta\}(n) = \frac{1}{2}(d_{1,3}(8n+7) - d_{2,3}(8n+7)),$
(1.15) $r\{\Delta + \Delta + \Delta + \Delta\}(n) = \sigma(2n+1),$
(1.16) $r\{\square + \square + \square + \square\}(n) = \sigma(4n+1).$

It should be noted that not all these results are new. For instance, (1.8) is equivalent to a result of Ramanujan ([1, pp.223-224, 3, 4, p.229]).

2. PRELIMINARY RESULTS

As usual, let

$$\phi(q) = \sum_{-\infty}^{\infty} q^{n^2}, \quad \psi(q) = \sum_{n \geq 0} q^{(n^2+n)/2}.$$

We shall require the easy lemmas

$$(2.1) \quad \phi(q)\psi(q^2) = \psi(q)^2,$$

$$(2.2) \quad \phi(q) = \phi(q^4) + 2q\psi(q^8),$$

as well as the (apparently new) result

$$(2.3) \quad \psi(q)\psi(q^3) = \phi(q^6)\psi(q^4) + q\phi(q^2)\psi(q^{12}).$$

Proofs of lemmas.

$$(2.1) \quad \phi(q)\psi(q^2) = \frac{(q^2)_{\infty}^5}{(q)_{\infty}^2 (q^4)_{\infty}^2} \cdot \frac{(q^4)_{\infty}^2}{(q^2)_{\infty}} = \frac{(q^2)_{\infty}^4}{(q)_{\infty}^2} = \psi(q)^2.$$

$$(2.2) \quad \phi(q) = \sum_{-\infty}^{\infty} q^{n^2} = \sum_{n \text{ even}} q^{n^2} + \sum_{n \text{ odd}} q^{n^2} = \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{4n^2+4n+1} = \phi(q^4) + 2q\psi(q^8).$$

$$\begin{aligned}
 (2.3) \quad q^4\psi(q^8)\psi(q^{24}) &= \sum_{k,l=-\infty}^{\infty} q^{(4k+1)^2+3(4l+1)^2} = \sum_{k,l=-\infty}^{\infty} q^{4(k+3l+1)^2+12(k-l)^2} \\
 &= \sum_{u-v \equiv 1 \pmod{4}} q^{4u^2+12v^2}.
 \end{aligned}$$

We now consider the two cases v even, u even. If v is even, $v = 2k$, $u = 4l + 1$ or $-4l - 1$, according as k is even or odd, while if u is even, $u = 2k$, $v = 4l + 1$ or $-4l - 1$, according as k is odd or even. Thus

$$\begin{aligned}
 \sum_{u-v \equiv 1 \pmod{4}} q^{4u^2+12v^2} &= \sum_{k,l=-\infty}^{\infty} q^{4(4l+1)^2+12(2k)^2} + \sum_{k,l=-\infty}^{\infty} q^{4(2k)^2+12(4l+1)^2} \\
 &= q^4\psi(q^{32})\phi(q^{48}) + q^{12}\phi(q^{16})\psi(q^{96}),
 \end{aligned}$$

as required.

Proofs of theorems.

(J1) is equivalent to

$$(3.1) \quad \phi(q)^2 = 1 + 4 \sum_{n \geq 1} (d_{1,4}(n) - d_{3,4}(n))q^n.$$

That is, by (2.2) and (2.1),

$$(3.2) \quad (\phi(q^4) + 2q\psi(q^8))^2 = (\phi(q^4)^2 + 4q^2\psi(q^8)^2) + 4q\psi(q^4)^2 = 1 + 4 \sum_{n \geq 1} (d_{1,4}(n) - d_{3,4}(n))q^n.$$

If we extract those terms in which the power of q is $1 \pmod{4}$, divide by $4q$ and replace q^4 by q , we find

$$(3.3) \quad \psi(q)^2 = \sum_{n \geq 0} (d_{1,4}(4n+1) - d_{3,4}(4n+1))q^n,$$

from which we obtain (1.1).

By (2.1), (3.3) can be written

$$(3.4) \quad \phi(q)\psi(q^2) = \sum_{n \geq 0} (d_{1,4}(4n+1) - d_{3,4}(4n+1))q^n,$$

from which (1.2) follows.

Using (2.2), (3.4) can be written

$$(3.5) \quad \psi(q^2)(\phi(q^4) + 2q\psi(q^8)) = \sum_{n \geq 0} (d_{1,4}(4n+1) - d_{3,4}(4n+1))q^n,$$

from which (1.3) and (1.4) follow.

(D) is equivalent to

$$(3.6) \quad \phi(q)\phi(q^2) = 1 + 2 \sum_{n \geq 1} (d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n))q^n,$$

or, by (2.2),

$$(3.7) \quad (\phi(q^4) + 2q\psi(q^8))(\phi(q^8) + 2q^2\psi(q^{16})) = 1 + 2 \sum_{n \geq 1} (d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n))q^n,$$

from which (1.5), (1.6) and (1.7) follow.

(L) is equivalent to

$$(3.8) \quad \begin{aligned} \phi(q)\phi(q^3) &= 1 + 2 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^n + 4 \sum_{n \geq 1} (d_{4,12}(n) - d_{8,12}(n))q^n \\ &= 1 + 2 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^n + 4 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^{4n}, \end{aligned}$$

or

$$(3.9) \quad \begin{aligned} (\phi(q^4) + 2q\psi(q^8)) (\phi(q^{12}) + 2q^3\psi(q^{24})) \\ = 1 + 2 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^n + 4 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^{4n}. \end{aligned}$$

If from (3.9) we extract those terms in which the power of q is $0 \pmod{4}$ and replace q^4 by q , we find

$$(3.10) \quad \begin{aligned} \phi(q)\phi(q^3) + 4q\psi(q^2)\psi(q^6) \\ = 1 + 2 \sum_{n \geq 1} (d_{1,3}(4n) - d_{2,3}(4n))q^n + 4 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^n \\ = 1 + 6 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^n \end{aligned}$$

where we have used the fact, which needs a little consideration, that

$$d_{1,3}(4n) - d_{2,3}(4n) = d_{1,3}(n) - d_{2,3}(n).$$

If we subtract (3.8) from (3.10), we find

$$(3.11) \quad 4q\psi(q^2)\psi(q^6) = 4 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^n - 4 \sum_{n \geq 1} (d_{1,3}(n) - d_{2,3}(n))q^{4n}.$$

The right hand side of (3.11) is an odd function of q ; if we divide by $4q$ and replace q^2 by q , we find

$$(3.12) \quad \psi(q)\psi(q^3) = \sum_{n \geq 0} (d_{1,3}(2n+1) - d_{2,3}(2n+1))q^n,$$

from which (1.8) follows.

If we invoke (2.3), (3.12) becomes

$$(3.13) \quad \phi(q^6)\psi(q^4) + q\phi(q^2)\psi(q^{12}) = \sum_{n \geq 0} (d_{1,3}(2n+1) - d_{2,3}(2n+1))q^n,$$

so

$$(3.14) \quad \phi(q^3)\psi(q^2) = \sum_{n \geq 0} (d_{1,3}(4n+1) - d_{2,3}(4n+1))q^n$$

and

$$(3.15) \quad \phi(q)\psi(q^6) = \sum_{n \geq 0} (d_{1,3}(4n+3) - d_{2,3}(4n+3))q^n.$$

(1.9) and (1.10) follow.

(3.14) and (3.15) can be written respectively

$$(3.16) \quad \psi(q^2) (\phi(q^{12}) + 2q^3\psi(q^{24})) = \sum_{n \geq 0} (d_{1,3}(4n+1) - d_{2,3}(4n+1))q^n$$

and

$$(3.17) \quad \psi(q^6) (\phi(q^4) + 2q\psi(q^8)) = \sum_{n \geq 0} (d_{1,3}(4n+3) - d_{2,3}(4n+3))q^n.$$

(1.11), (1.12), (1.13) and (1.14) follow.

(J2) is equivalent to

$$(3.18) \quad \phi(q)^4 = 1 + 8 \sum_{n \geq 1} \left(\sum_{d|n, 4 \nmid d} d \right) q^n.$$

Now, the left hand side is

$$(3.19) \quad \begin{aligned} \phi(q)^4 &= (\phi(q^4) + 2q\psi(q^8))^4 \\ &= (\phi(q^4)^4 + 16q^4\psi(q^8)^4) + 8q\phi(q^4)^3\psi(q^8) + 24q^2\phi(q^4)^2\psi(q^8)^2 + 32q^3\phi(q^4)\psi(q^8)^3 \\ &= (\phi(q^4)^4 + 16q^4\psi(q^8)^4) + 8q\phi(q^4)^2\psi(q^4)^2 + 24q^2\psi(q^4)^4 + 32q^3\psi(q^4)^2\psi(q^8)^2. \end{aligned}$$

So (3.18) becomes

$$(3.20) \quad \begin{aligned} &(\phi(q^4)^4 + 16q^4\psi(q^8)^4) + 8q\phi(q^4)^2\psi(q^4)^2 + 24q^2\psi(q^4)^4 + 32q^3\psi(q^4)^2\psi(q^8)^2 \\ &= 1 + 8 \sum_{n \geq 1} \left(\sum_{d|n, 4 \nmid d} d \right) q^n. \end{aligned}$$

We deduce that

$$(3.21) \quad 24\psi(q)^4 = 8 \sum_{n \geq 0} \left(\sum_{d|4n+2} d \right) q^n = 24 \sum_{n \geq 0} \sigma(2n+1)q^n$$

and

$$(3.22) \quad 8\phi(q)^2\psi(q)^2 = 8 \sum_{n \geq 0} \left(\sum_{d|4n+1} d \right) q^n = 8 \sum_{n \geq 0} \sigma(4n+1)q^n,$$

which are (1.15) and (1.16).

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