ALGEBRAIC SUCCESSION RULES

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ABSTRACT. In this paper, we show that succession rules

$$(k) \rightsquigarrow (1)^{\alpha_{k-1}} \dots (k-1)^{\alpha_1} (k)^{\lambda_0} \dots (k+p)^{\lambda_p}$$

have an algebraic generating function when the sequence (α_i) is rational. We decompose algebraically the paths in the corresponding generating tree and deduce an algebraic equation satisfied by the noncommutative generating function.

RÉSUMÉ. Nous montrons dans cet article que les règles de succession

$$(k) \rightsquigarrow (1)^{\alpha_{k-1}} \dots (k-1)^{\alpha_1} (k)^{\lambda_0} \dots (k+p)^{\lambda_p}$$

ont une série génératrice algébrique quand la suite (α_i) est rationnelle. Nous décomposons algébriquement les chemins dans l'arbre de génération correspondant et nous en déduisons une équation algébrique satisfaite par la série génératrice non commutative.

1. INTRODUCTION

The succession rules were first introduced by Chung, Graham, Hoggatt and Kleimann in [3] to study Baxter permutations. The method was later successfully used by West [11], Dulucq, Gire and Guibert [5, 6, 7] for the enumeration of permutations with forbidden sequences. The concept has more recently been exploited by Barcucci, Del Lungo, Pergola and Pinzani [2] as the ECO method for the enumeration and recursive construction of various classes of combinatorial objects. The succession rule approach has several equivalent interpretations, ECO rules, random paths, infinite automata or Riordan arrays, and deals with different kinds of generating functions (rational, algebraic or exponential). The problem of classifying successions rules according to the type of their generating functions has been proposed by R.Pinzani [2] in the area of ECO systems. A classical and easy result is that finite succession rules have rational generating function since they correspond to a regular language. It is shown in [1] that every finite transformation of Catalan succession rule

$$(k) \rightsquigarrow (1)(2) \dots (k)(k+1)$$

is algebraic. In the same paper are also described succession rules leading to exponential generating functions which have been more extensively studied by S.Corteel in [4].

Our paper is devoted to the study of algebraic system of succession rules having algebraic generating function. Our approach is closely related with the Schutzenberger methodology, which consists in finding first a bijection between the objects and the words of an algebraic language, second a non ambiguous grammar for the language and finally take the commutative image and deduce an algebraic system for the generating function. We first explore the basic Catalan example and explain its algebraicity using a non ambiguous decomposition of the paths in the Catalan generating tree. For that, we define its noncommutative formal power series using the infinite alphabet of positive integers. We use a new operation $\oplus$ which allows us to get a non ambigous decomposition of the formal power series associated to the generating tree. We deduce the classical Catalan algebraic equation by taking the commutative image of the formal power series. This method is then extended to the more general succession rule

$$(k) \rightsquigarrow (1)\dots(k-1)(k)^{\lambda_0}\dots(k+p)^{\lambda_p}$$

for any finite sequence (λ_i).

Next we describe an algebraic decomposition for the succession rule

$$(k) \rightsquigarrow (1)^{\alpha_{k-1}}\dots(k-1)^{\alpha_1}(k)^{\lambda_0}\dots(k+p)^{\lambda_p}$$

for any *constant* sequence (α_i) proving thereby that their generating function is algebraic when the sequence (α_i) is *rational.*

2. Definitions

A succession rule is a function $(k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_{s_k}(k))$ which associates to each positive integer k a finite multiset of integers, called successors of k. A generating tree is a succession rule with a particular integer a, called axiom. We suppose in the following that a equals 1. A generating tree can be viewed as the infinite tree constructed with a root labelled by the axiom and where each node labelled k has sons labelled according to the succession rule. Hence, ECO systems are those generating trees where each integer has exactly k successors.

For a generating tree Ω,

$$\Omega \left\{ \begin{array}{l} (1) \\ (k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_{s_k}(k)) \end{array} \right. \tag{1}$$

we define the language L_Ω as the set of words over $\mathbb{N}$, begining by the axiom 1 and satisfying the succession rule, $(k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_{s_k}(k))$. Each word w of L_Ω corresponds to at least one path of Ω. We note $m(w)$ the number of paths w in the generating tree Ω. We denote by S_Ω the noncommutative formal power series,

$$S_\Omega = \sum_{w\in L_\Omega} m(w)w.$$

By construction, the generating tree Ω and the noncommutative formal power series S_Ω have the same generating function,

$$F_\Omega(z) = \sum_n f_n z^n$$

where

$$f_n = \sum_{w\in L_\Omega, |w|=n} m(w).$$

For simplifying the notation, we write F_Ω for $F_\Omega(z)$. We use standard external product and concatenation, by an integer n, over the noncommutative formal power series S_Ω. We write:

$$nS_\Omega = \sum_{w\in L_\Omega} (nm(w))w$$

$$(n).S_\Omega = \sum_{w\in L_\Omega} m(w)(n.w).$$

We define a new operation $\oplus$ as follows,

Definition 1. *For $i \in \mathbb{N}^+$, we define by $i^\oplus = i+1$. By extension if $w = w_1w_2\dots w_i$ is a word of L_Ω then $w^\oplus = w_1^\oplus w_2^\oplus \dots w_i^\oplus$ and $S_\Omega^\oplus = \sum_{w\in L_\Omega} m(w)w^\oplus$*

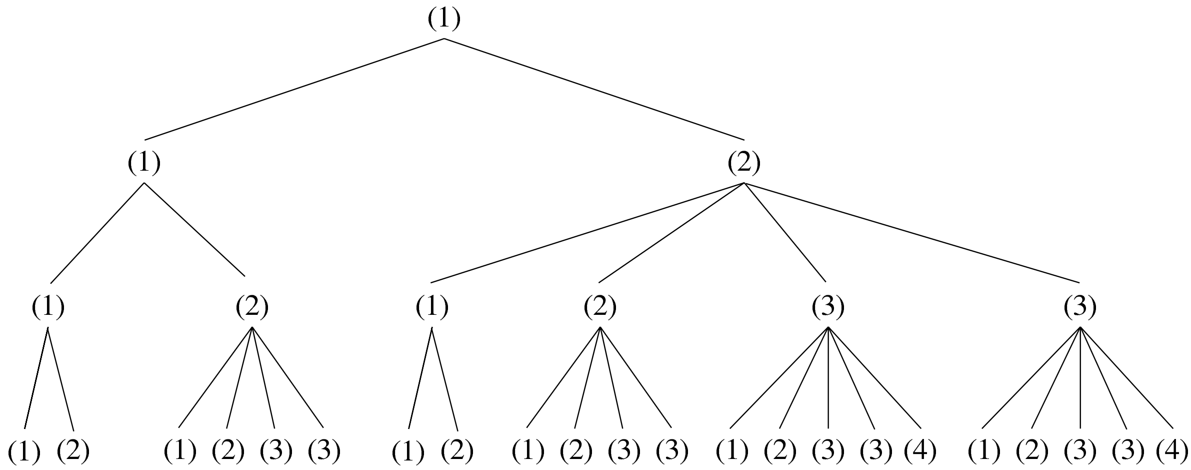

FIGURE 1. Partial generating tree of the rule Θ

Example 2. *The Figure 1 give the partial generating tree of the rule* Θ *with axiom* 1.

$$\Theta \left\{ \begin{array}{l} (1) \\ (k) \rightsquigarrow (1)(2)(3)(3)(4)\dots(k)(k+1) \end{array} \right.$$

$$\begin{aligned} S_\Theta &= (1)+(11)+(12)+(111)+(112)+(121)+(122)+2(123)+\dots \\ F_\Theta &= z+2z^2+6z^3+22z^4+\dots \end{aligned}$$

The noncommutative formal power series approach allows us to interpret finite transformations and show that they do not change the algebraicity of the generating function. We are interested only with *total* algebraic succession rules, that is succession rules where algebraicity is acquired for any choice of the axiom, and for all the restrictions according to the last letter. More precisely, we suppose that for any integer i, the generating function F_i of the paths ending with i are algebraic.

Definition 3. *The transformation* T_1 *is the addition of a constant* c *for one succession rule,*

$$T_1(\Omega) \left\{ \begin{array}{l} (1) \\ (k_0) \rightsquigarrow (e_1(k_0))(e_2(k_0))\dots(e_{s_{k_0}}(k_0))(c) \\ (k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_{s_k}(k)), for\ \ k \neq k_0 \end{array} \right.$$

Definition 4. *The transformation* T_2 *is the addition of a constant* c *uniformly,*

$$T_2(\Omega) \left\{ \begin{array}{l} (1) \\ (k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_{s_k}(k))(c) \end{array} \right.$$

Proposition 5. *All finite modifications by* T_1 *and* T_2 *of the succession rule* Ω *are algebraic.*

Proof. Let S_{k_0} be the formal sum of the words corresponding to the paths of the generating tree Ω ending with k_0, let F_{k_0} the generating function of S_{k_0}. We have $S_{T_1(\Omega)} = S_\Omega + S_{k_0}S_{T_1(\Omega)}$, and deduce $F_{T_1(\Omega)} = F_\Omega + F_{k_0}F_{T_1(\Omega)}$, so $F_{T_1(\Omega)}$ is algebraic when F_Ω and F_{k_0} are algebraic.
Let now ${}_cS$ be the formal power series of

$$\left\{ \begin{array}{l} (c) \\ (k) \rightsquigarrow (e_1(k))(e_2(k))\dots(e_{s_k}(k)) \end{array} \right.$$

that is the succession rule Ω where the axiom 1 has been replaced by c. Let ${}_cF$ the generating function of ${}_cS$. We have $S_{T_2(\Omega)} = S_\Omega({}_cS)^* = \frac{S_\Omega}{1-({}_cS)}$, which concludes the proof. □

Example 6. *Let Γ be the Catalan generating tree defined by,*

$$\Gamma \left\{ \begin{array}{l} (1) \\ (k) \rightsquigarrow (1)(2)\dots(k)(k+1) \end{array} \right.$$

the Θ generating tree of Example2 is obtained applying the T_2 transformation over Γ with $c = 3$.

Remark Proposition 5 will be usefull in Theorem 12.

3. Algebraicity of Catalan generating tree Γ

In this section, we first describe a non ambiguous decomposition of the Catalan generating tree, which explains its algebraicity.

Theorem 7. *The formal power series S_Γ satisfies the equation*

$$S_\Gamma = (1) + (1).S_\Gamma + (1).S_\Gamma^\oplus.(\epsilon + S_\Gamma)$$

Proof. The proof can be easily deduced from the following non ambiguous inductive description of the set of the paths in the generating tree. Let $w \neq 1$ be a non trivial path of the generating tree Γ. Then w can be written $w = 1u$ where either u begins with 1 and therefore is a path of Γ or u begins with 2 and we have two cases,

- if each letter of u is > 1 then $u = v^\oplus$ where v is a path of Γ,
- if not, u can be writen $v_1^\oplus v_2$ where v_1 and v_2 are paths of Γ, v_2 being the longest suffix of u beginning by 1.

We deduce that $S_\Gamma = (1) + (1).S_\Gamma + (1).S_\Gamma^\oplus + (1).S_\Gamma^\oplus.S_\Gamma$ □

Corollary 8. *The generating function F_Γ of the succession rule Γ satisfies:*

$$F_\Gamma = z + zF_\Gamma + zF_\Gamma(1 + F_\Gamma)$$

Theorem 7 can be easily generalized to the case of more general succession rule,

$$\Lambda \left\{ \begin{array}{l} (1) \\ (k) \rightsquigarrow (1)\dots(k-1)(k)^{\lambda_0}\dots(k+p)^{\lambda_p} \end{array} \right.$$

A similar inductive description (see Figure 2) of the associated language leads to the following theorem.

Theorem 9.

$$\begin{array}{rcl} S_\Lambda & = & (1) + \\ & & \lambda_0(1).S_\Lambda + \\ & & \lambda_1(1).S_\Lambda^\oplus.(\epsilon + S_\Lambda) + \\ & & \lambda_2(1).S_\Lambda^{\oplus\oplus}.(\epsilon + S_\Lambda^\oplus).(\epsilon + S_\Lambda) + \\ & & \vdots \\ & & \lambda_p(1).S_\Lambda^{p\oplus}.(\epsilon + S_\Lambda^{(p-1)\oplus}).\dots.(\epsilon + S_\Lambda^\oplus)(\epsilon + S_\Lambda), \end{array}$$

where $S_\Lambda^{i\oplus} = (S_\Lambda^{(i-1)\oplus})^\oplus$ and $S_\Lambda^{1\oplus} = S_\Lambda^\oplus$.

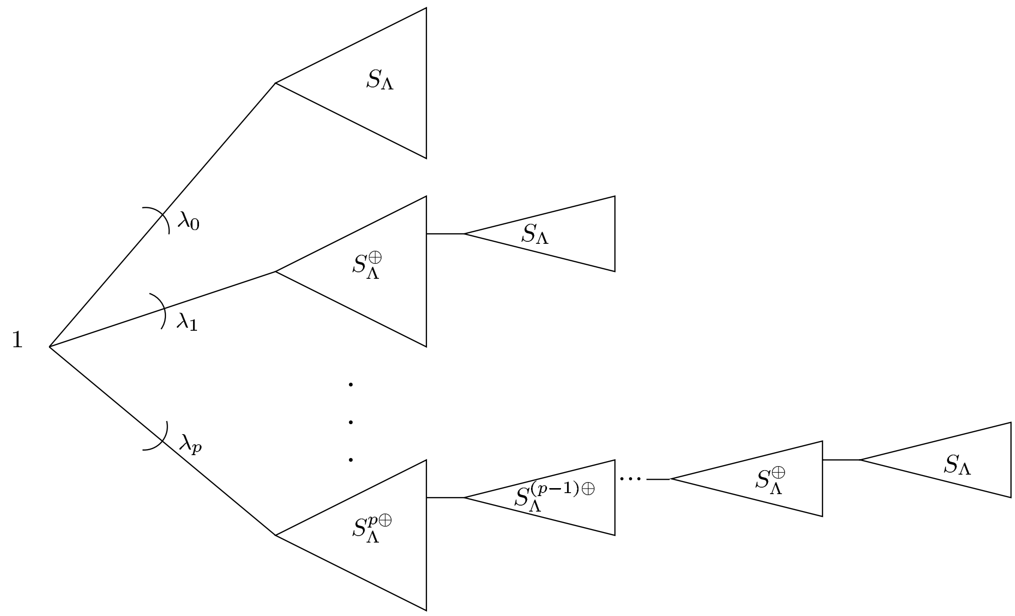

FIGURE 2. Generalized Catalan generating tree Λ

Corollary 10. *The generating function F_Λ of the succession rule Λ is algebraic and satisfies :*

$$\begin{aligned} F_\Lambda &= z+ \\ &\quad \lambda_0 z F_\Lambda + \\ &\quad \lambda_1 z F_\Lambda(1+F_\Lambda)+ \\ &\quad \lambda_2 z F_\Lambda(1+F_\Lambda)^2+ \\ &\quad \vdots \\ &\quad \lambda_p z F_\Lambda(1+F_\Lambda)^p \\ &= z(1+\sum_{i=0}^{p}\lambda_i F_\Lambda(1+F_\Lambda)^i) \end{aligned}$$

All the algebraic generating function given in the small catalog of ECO-systems of [1] can be deduced from the previous theorem. For instance, Motzkin numbers correspond to the sequence $\lambda_i = (0,1,0,\ldots)$, Schröder numbers correspond to the sequence $\lambda_i = (1,2,0,\ldots)$ and Ternary trees correspond to the sequence $\lambda_i = (1,1,1,0,\ldots)$.

4. ALGEBRAICITY AND RATIONALITY

In this section, we study more general rules having the following general form,

$$\Upsilon \left\{ \begin{array}{l} (1) \\ (k) \rightsquigarrow (1)^{\alpha_{k-1}} \ldots (k-1)^{\alpha_1}(k)^{\lambda_0}\ldots(k+p)^{\lambda_p} \end{array}\right.$$

As in the previous section, the words occuring in S_Υ have λ_i rises from (k) to $(k+i)$. The difficulty is to deal here with the α_i kind of descents from (k) to $(k-i)$.

Theorem 11. *The rationality of the sequence (α_i) implies the algebraicity of F_Υ.*

Proof. We will give the proof in the particular case of $p = 2$ and (α_i) a rational sequence of degree 2. The general case follows the same scheme. We begin by giving the different equations obtained from the recursive decomposition of the paths in the generating tree Υ. We need to define S_i as the formal sum of the paths in Υ ending by i:

$$\begin{aligned}
S &= (1) + \lambda_0(1)S + \lambda_1(1)S^{\oplus} + \lambda_1(1)\sum_{j\geq 1}\alpha_j S_j^{\oplus} S \\
&\quad \lambda_2(1)S^{\oplus\oplus} + \lambda_2(1)\sum_{j\geq 1}\alpha_{j+1}S_j^{\oplus\oplus}S + \lambda_2(1)\sum_{j\geq 1}\alpha_j S_j^{\oplus\oplus}S^{\oplus} \\
&\quad +\lambda_2(1)\sum_{j\geq 1}\sum_{k\geq 1}\alpha_j\alpha_k S_j^{\oplus\oplus}S_k^{\oplus}S \\
S_1 &= (1) + \lambda_0(1)S_1 + \lambda_1(1)\sum_{j\geq 1}\alpha_j S_j^{\oplus}S_1 \\
&\quad +\lambda_2(1)\sum_{j\geq 1}\alpha_{j+1}S_j^{\oplus\oplus}S_1 \\
&\quad +\lambda_2(1)\sum_{j\geq 1}\sum_{k\geq 1}\alpha_j\alpha_k S_j^{\oplus\oplus}S_k^{\oplus}S_1 \\
S_2 &= \lambda_0(1)S_2 + \lambda_1(1)S_1^{\oplus} + \lambda_1(1)\sum_{j\geq 1}\alpha_j S_j^{\oplus}S_2 \\
&\quad +\lambda_2(1)\sum_{j\geq 1}\alpha_{j+1}S_j^{\oplus\oplus}S_2 + \lambda_2(1)\sum_{j\geq 1}\alpha_j S_j^{\oplus\oplus}S_1^{\oplus} \\
&\quad +\lambda_2(1)\sum_{j\geq 1}\sum_{k\geq 1}\alpha_j\alpha_k S_j^{\oplus\oplus}S_k^{\oplus}S_2
\end{aligned}$$

and for $i \geq 3$,

$$\begin{aligned}
S_i &= \lambda_0(1)S_i + \lambda_1(1)S_{i-1}^{\oplus} + \lambda_1(1)\sum_{j\geq 1}\alpha_j S_j^{\oplus}S_i \\
&\quad \lambda_2(1)S_{i-2}^{\oplus\oplus} + \lambda_2(1)\sum_{j\geq 1}\alpha_{j+1}S_j^{\oplus\oplus}S_i + \lambda_2(1)\sum_{j\geq 1}\alpha_j S_j^{\oplus\oplus}S_{i-1}^{\oplus} \\
&\quad +\lambda_2(1)\sum_{j\geq 1}\sum_{k\geq 1}\alpha_j\alpha_k S_j^{\oplus\oplus}S_k^{\oplus}S_i
\end{aligned}$$

We note $F = F_\Upsilon$ for short, the generating function of Υ, F_i the generating functions of S_i, and

$$F = \sum_{i\geq 1} F_i$$

$$G = \sum_{i\geq 1} \alpha_i F_i$$

$$H = \sum_{i\geq 1} \alpha_{i+1} F_i$$

Thus we have,

$$\begin{aligned}
F &= z + \lambda_0 zF + \lambda_1 zF + \lambda_1 zGF + \lambda_2 zF + \lambda_2 zHF + \lambda_2 zGF + \lambda_2 zG^2F \\
F_1 &= z + \lambda_0 zF_1 + \lambda_1 zGF_1 + \lambda_2 zHF_1 + \lambda_2 zG^2F_1 \\
F_2 &= \lambda_0 zF_2 + \lambda_1 zF_1 + \lambda_1 zGF_2 + \lambda_2 zHF_2 + \lambda_2 zGF_1 + \lambda_2 zG^2F_2
\end{aligned}$$

and for $i \geq 3$,

$$F_i = \lambda_0 zF_i + \lambda_1 zF_{i-1} + \lambda_1 zGF_i + \lambda_2 zF_{i-2} + \lambda_2 zHF_i + \lambda_2 zGF_{i-1} + \lambda_2 zG^2F_i$$

The generating functions F_i satisfies a linear recurence relation of degree 2, $F_{i+2} = uF_{i+1} + vF_i$ for $i \geq 1$, where u and v are rational functions depending on $\lambda_0, \lambda_1, \lambda_2, G$ and H.

Moreover, supposing that the sequence (α_i) is rational of degree 2 means that it satisfies a recurence relation $\alpha_{i+2} = a\alpha_{i+1} + b\alpha_i + c$, for some a, b, c coefficients. Developing $\sum_{i\geq 1} \alpha_{i+2}M^i$, we get,

$$\begin{aligned}
\sum_{i\geq 1} \alpha_{i+2}M^i &= a\sum_{i\geq 1} \alpha_{i+1}M^i + b\sum_{i\geq 1} \alpha_i M^i + c\sum_{i\geq 1} M^i \\
&= a\alpha_2 M + b\alpha_1 M + b\alpha_2 M^2 + +c\sum_{i\geq 1} M^i + (aM + bM^2)\sum_{i\geq 1} \alpha_{i+2}M^i
\end{aligned}$$

where in particular, M can be any square matrix.
Let now

$$M = \begin{pmatrix} u & v \\ 1 & 0 \end{pmatrix},$$

where u and v are the coefficients of the linear recurrence satisfied by F_i. Classicaly, we have

$$\begin{pmatrix} F_{i+2} \\ F_{i+1} \end{pmatrix} = M \begin{pmatrix} F_{i+1} \\ F_i \end{pmatrix} = M^2 \begin{pmatrix} F_i \\ F_{i-1} \end{pmatrix} = \ldots = M^i \begin{pmatrix} F_2 \\ F_1 \end{pmatrix},$$

and we obtain

$$\begin{aligned}
\sum_{i\geq 1} \alpha_{i+2} \begin{pmatrix} F_{i+2} \\ F_{i+1} \end{pmatrix} &= (a\alpha_2 M + b\alpha_1 M + b\alpha_2 M^2) \begin{pmatrix} F_2 \\ F_1 \end{pmatrix} + c\sum_{i\geq 1} M^i \begin{pmatrix} F_2 \\ F_1 \end{pmatrix} \\
&\quad + (aM + bM^2)\sum_{i\geq 1} \alpha_{i+2} \begin{pmatrix} F_{i+2} \\ F_{i+1} \end{pmatrix}
\end{aligned}$$

As $\sum_{i\geq 1} \alpha_{i+2}F_{i+2} = G - \alpha_1 F_1 - \alpha_2 F_2$ and $\sum_{i\geq 1} \alpha_{i+2}F_{i+1} = H - \alpha_2 F_1$, we deduce an algebraic system of equations satisfied by G and H,

$$\begin{aligned}
\begin{pmatrix} G \\ H \end{pmatrix} &= (a\alpha_2 M + b\alpha_1 M + b\alpha_2 M^2) \begin{pmatrix} F_2 \\ F_1 \end{pmatrix} + c\begin{pmatrix} F - F_1 - F_2 \\ F - F_1 \end{pmatrix} \\
&\quad + (aM + bM^2)\begin{pmatrix} G - \alpha_1 F_1 - \alpha_2 F_2 \\ H - \alpha_2 F_1 \end{pmatrix}
\end{aligned}$$

□

5. Conclusion

Theorem 11 and Proposition 5 allow us to generalize the results of Flajolet and al [1]:

Theorem 12. *All finite transformations of the succession rule*

$$(k) \rightsquigarrow (1)^{\alpha_{k-1}} \dots (k-1)^{\alpha_1} (k)^{\lambda_0} \dots (k+p)^{\lambda_p}$$

are algebraic when (α_i) *is rational.*

A conjecture is to have a similar result when the sequence (α_i) is algebraic as discussed with Cyril Banderier during GASCOM'01.

We give some examples where such succesion rules naturally appear (see Figure 3). Diagonally directed convex polyominoes [14] (or fully directed compact animals) are known to be counted according to their number of diagonals by $\frac{1}{2n+1}\binom{3n}{n}$. They naturally satisfy the succession rule

$$\begin{cases} (1) \\ (k) \rightsquigarrow (1)^{k+1}(2)^k \dots (k-1)^3 (k)^2 (k+1) \end{cases}$$

Another example concerns a new succesion rule for Catalan numbers.

$$\begin{cases} (1) \\ (k) \rightsquigarrow (1)^{2^{k-2}}(2)^{2^{k-3}} \dots (k-2)^2 (k-1)(k+1) \end{cases}$$

This succession rule generates the partition $\{B_1, \dots B_p\}$ of $[n]$ such that the numbers $1, 2, \dots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, \dots, B_p$ are pairwise disjoint [13]. Indeed, let k be the number of isolated points around 1. The 2^{k-1} successors of this configuration are obtained by taking all the subset of $\{\alpha_1 = 1, \alpha_2, \dots, \alpha_k, n+1\}$ containing $n+1$.

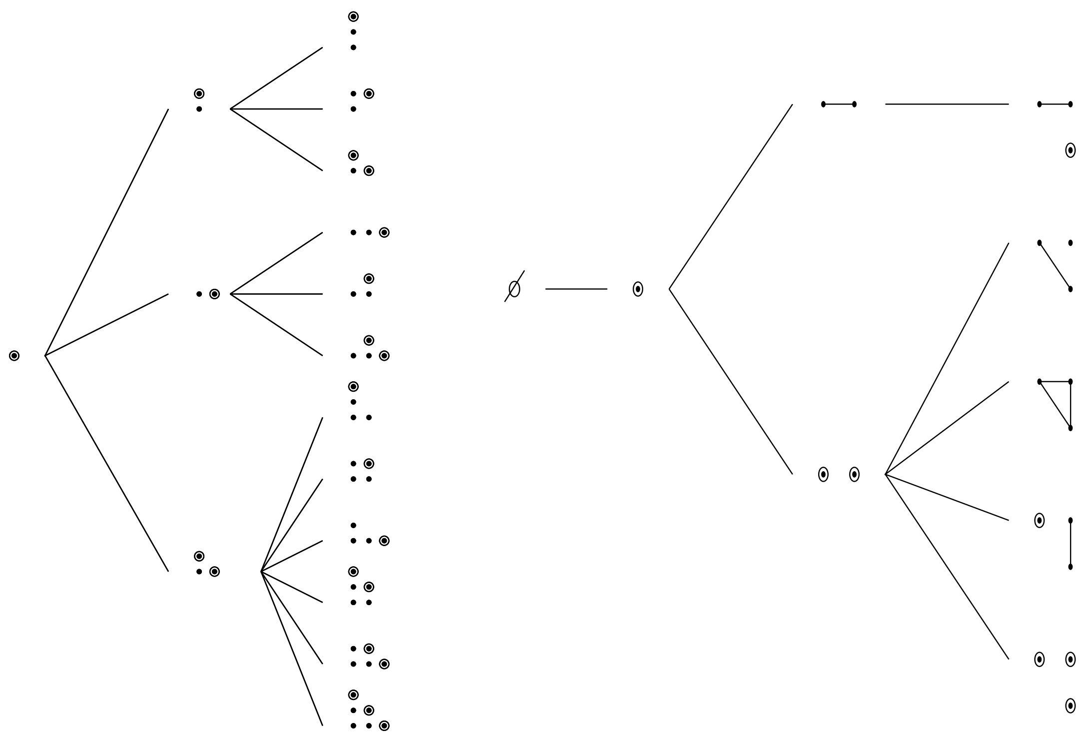

FIGURE 3. Generating trees for FDC animals and Catalan blocks

REFERENCES

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Mathematics*, 2000. An earlier version "On Generating Functions of Generating Trees" was in the proceedings of FPSAC'99 and as INRIA Report 3661.

[2] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani, ECO: A methodology for the Enumeration of Combinatorial Objects, *Journal of Difference Equations and Applications,* 5 (1999) 435-490.

[3] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, M. Kleimann, The number of Baxter permutations, *J. Combin. Theory Ser. A,* 24 (1978) 382-394.

[4] S. Corteel. *Séries génératrices exponentielles pour les eco-systèmes signés,* Proceedings of the 12-th International Conference on Formal Power Series and Algebraic Combinatorics, Moscow, 2000.

[5] S. Dulucq, S. Gire, J. West Permutations with forbidden subsequences and nonseparable planar maps, *Discrete Mathematics,* 153 (1996), 85-103.

[6] S. Dulucq, S. Gire, O. Guibert, A combinatorial proof of J.West's conjecture, *Discrete Mathematics,* 187 (1998), 71-96.

[7] S. Dulucq, S. Gire, O. Guibert, Enumération de permutations à motifs exclus, *Actes du 30ième séminaire Lotharingien de Combinatoire, Hesselberg,* Janvier 1993.

[8] D. Merlini and M.C. Verri, Generating trees and proper Riordan Arrays, *Discrete Mathematics,* 218 (2000), 167-183.

[9] D. Merlini, R. Sprugnoli and M.C. Verri, An algebra for Proper Generating Trees, *Proceedings of CMCS, Versailles* (2000), 127-139

[10] E. Pergola, R. Pinzani and S. Rinaldi, A set of well-defined operations on succession rules *Proceedings of CMCS, Versailles* (2000), 141-152.

[11] J. West, Permutations with forbidden subsequences and stack-sortable permutations, *Ph.D. Thesis, MIT, Cambridge, MA* (1990).

[12] J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Mathematics,* 146 (1995), 247-262.

[13] R P. Stanley, Exercises on Catalan and Related Numbers, *Enumerative Combinatorics,* 2 (1999), 221-247.

[14] S. Feretić, A q-Enumeration of Directed Diagonally Convex Polyominoes, *FPSAC'99,* (1999), 195-206.

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