SKEW OSCILLATING SEMI-STANDARD TABLEAUX (EXTENDED ABSTRACT)

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ABSTRACT. We introduce an analogue of the Robinson-Schensted correspondence for skew oscillating semi-standard tableaux which generalize the correspondence for skew oscillating tableaux. We give the geometric construction for skew oscillating semi-standard tableaux and examine its combinatorial properties.

RÉSUMÉ. Nous introduisons une construction analogue de la correspondance de Robinson-Schensted pour les tableaux semi-standards oscillants gauches qui généralise la correspondance pour les tableaux oscillants gauches. Nous donnons la construction géométrique pour les tableaux semi-standards oscillants gauches et exeminons ses propriétés combinatoires.

1. INTRODUCTION

The Robinson-Schensted correspondence between permutations and pairs of standards tableaux of the same shape is introduced by Robinson ([6]) and it is given by Schensted ([10]) a little different form. After generalization by Knuth ([5]) to generalized permutations and pairs of semi-standards tableaux, various analogues of the Robinson-Schensted correspondence have been produced on different kinds of tableaux ([8],[13],[3],[9]).

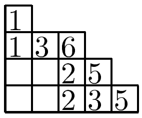
More recently, Dulucq and Sagan ([4]) have given the Robinson-Schensted correspondence for oscillating tableaux and skew oscillating tableaux.

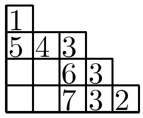
In this article, we extended the properties and constructions of analogue of Robinson-Schensted correspondence in [4] to skew oscillating semi-standard tableaux. In sections 2, we give basic definitions of generalised biwords and skew oscillating semi-standard tableaux. An algorithm of Robinson-Schensted for skew oscillating semi-standard tableaux is given in section 3, which is an extension of the algorithm of Robinson-Schensted correspondence for skew oscillating standard tableaux given in [4]. Then we give a geometric construction of a generalized biword due to Viennot, Chauve and Dulucq ([1],[8],[13]).

2. DEFINITION AND NOTATIONS

Let $\lambda = (\lambda_1, ..., \lambda_k)$, $\lambda_1 \leq ... \leq \lambda_k$, be a partition of n such that $\sum_{i=1}^{k} \lambda_i = n$. The partition λ can be displayed a Ferrers diagram with the part λ_i in the row i. If $\mu \subseteq \lambda$ then the corresponding skew shape λ/μ is the set $\{c | c \in \lambda, c \notin \mu\}$. If $|\lambda/\mu| = n$ then we write $\lambda/\mu \vdash n$ and say that λ/μ is a skew partition of n. A skew semi-standard tableau S of shape λ/μ is a labeling of the cells of λ/μ with positive integers so that the rows are strictly increasing and the columns are weakly increasing. $\emptyset_\alpha$ denotes the empty tableaux of the shape α (or a skew tableau of the shape α/α). $T(\lambda/\mu)$ denotes the set of skew semi-standard tableaux of shape λ/μ.

$S(i,j)$ denotes the label of the cell in the i^{th} row and j^{th} column of a skew semi-standard tableau S so that $k \in S$ means $k = S(i,j)$ for some i, j. $\overline{T}(\lambda/\mu)$ denotes the set of tableaux of shape λ/μ with rows strictly decreasing and columns weakly decreasing. For example, when $\lambda = (5,4,3,1)$ $\mu = (2,2)$, the two following tableaux belong to $T(\lambda/\mu)$ and $\overline{T}(\lambda/\mu)$ respectively.





Four kinds of insertions and deletions in a skew semi-stanaard tableau ([1],[4]) are defined below. Let S be a skew semi-standard tableau of shape λ/μ.

1. The *external insertion* inserts an integer x in S by using the Knuth-Robinson-Schensted algorithm([2],[5]). We denote the new tableau obtained after this insertion by $ExtI(S,x)$. The inverse process is called *external deletion*, denoted by $ExtD(S,x)$, which ends with the expulsion of an integer out of S.

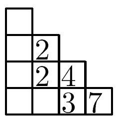
2. The *internal insertion* occurs only in a cell (u,v) of S such that $(u,v) \notin \mu$ and it belongs to one of three cases: (i) $(u-1,v) \in \mu$ and $(u,v-1) \in \mu$, (ii) $v = 1$ and $(u-1,v) \in \mu$, (iii) $u = 1$ and $(u,v-1) \in \mu$.

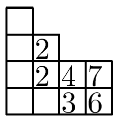
The *internal insertion* of the cell (u,v) inserts the integer x contained in $S(u,v)$ from the row $u+1$ using the external insertion algorithm. We denote the new tableau by $IntI(S,(u,v))$. The external deletion is called *internal deletion* if it ends in filling a cell of μ. $IntD(S,(u,v))$ denotes the internal deletion.

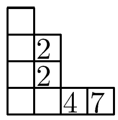
3. The *empty insertion* adds an empty cell (u,v) in S such that $(u,v) \notin \lambda$, satisfying (i)$(u-1,v) \in \mu$ and $(u,v-1) \in \mu$, (ii)$u = 1$, $(u,v-1) \in \mu$ or (iii)$v = 1$, $(u-1,v) \in \mu$. $EmpI(S,(u,v))$ denotes the new tableau obtained after this insertion and the inverse process is called *empty deletion*, denoted by $EmpD(S,(u,v))$.

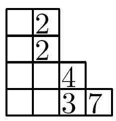
4. A cell can simply be *attached* or *erased* using neither the insertion algorithms nor the deletion algorithms.

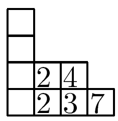
Example 2.1.

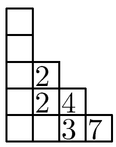
P = 

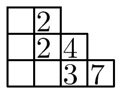
ExtI(P,6) = 

ExtD(P,3) = 

IntI(P,(2,2)) = 

IntD(P,(3,2)) = 

EmpI(P,(5,1)) = 

EmpD(P,(4,1)) = 

A skew oscillating semi-standard tableau of length n is a sequence of semi-standard tableaux $P = (P_0, P_1, ..., P_n)$ where P_k is obtained from P_{k-1} by an insertion or a deletion of a cell.

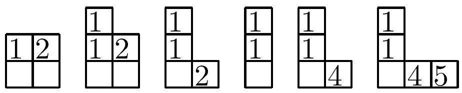
$\Theta_n(\alpha/\gamma \to \beta/\mu)$ denotes the set of skew oscillating tableaux $P = (P_0, P_1, ..., P_n)$ of length n satisfying the following conditions:

(1) the shape of P_0 is α/γ, and the shape of P_n is β/μ,

(2) P_k is obtained from P_{k-1} by attaching a cell with a label (this is not by the insertion algorithms) or a deletion of a cell by external deletion, internal deletion or empty deletion.

(3) if $x_i, x_j, ..., x_m$ are inserted respectively in $P_i, P_j, ..., P_m$, $i < j < ... < m$, then $x_i \le x_j \le ... \le x_m$.

For example, if $\alpha = (2, 2)$, $\gamma = (2)$, $\beta = (3, 1, 1)$ and $\mu = (1)$ then the following tableau belongs to $\Theta_5(\alpha/\gamma \to \beta/\mu)$. 1 is inserted in P_1, 4 in P_4 and 5 in P_5.



For a $P \in \Theta_n(\alpha/\gamma \to \beta/\mu)$, we define a set of nondecreasing sequences of positive integers $I(P) = \cup_{j \in N} I_j$, where $I_j = \{j_0, j_1, j_2, ..., j_n\}$, $j_0 = 0 \le j_1 \le ... \le j_n$ and $j_k = x$ if $P_k = P_{k-1} + (u, v)$ with $P_k(u, v) = x$ for $1 \le k \le n$. An I_j of the example above is $\{0, 1, j_2, j_3, 4, 5\}$, where j_2, j_3 are positive integers.

A skew oscillating semi-standard tableau of $\Theta(\emptyset\mu \to \lambda/\mu)$ having only insertion steps, is a skew semi-standard tableau of shape λ/μ, the label of a cell being given by its creation.

A generalized biword π of size $2n$ is a sequence of vertical pairs of positive integers $\pi = \begin{pmatrix} u_1 & u_2 & ... & u_n \\ v_1 & v_2 & ... & v_n \end{pmatrix}$ where $u_1 \ge u_2 \ge ... \ge u_k$, $u_i \ge v_i$ for $i = 1, ..., k$, and $v_i \ge v_2$ if $u_i = u_j$. $\hat{\pi}$ denotes the

top row of π and $\check{\pi}$ its bottom row.

GB denotes the set of generalized biwords. The size of π is $2\times$ the number of pairs of $\binom{u_i}{v_i}$, or $|\pi| = 2n$. GB_{2n} denotes the set of generalized biwords of size $2n$.

3. Generalized biwords and skew oscillating semi-standard tableaux

We give a description of an algorithm to examine the relation between skew oscillating semi-standard tableaux and the triples $(S, U, \pi) \in \cup_{\mu \subseteq \alpha \cap \beta} [T(\beta/\mu) \times \overline{T}(\alpha/\mu)] \times GB$.

Algorithm **OSCIL**

(a)The input is $(S, U, \pi) \in \cup_{\mu \subseteq \alpha \cap \beta} [T(\beta/\mu) \times \overline{T}(\alpha/\mu)] \times GB$,

(b) The output is (P, I) where $P \in \Theta_n(\emptyset_\alpha \to \beta/\mu)$, and $I = \{i_0 = 0, i_1, i_2, ..., i_n\} \in I(P)$, i.e., I satisfies the following conditions :

(1) $i_1, ..., i_n$ being a nondecreasing sequence of positive integers,

(2) if we obtain P_k from P_{k-1} by attaching a cell (u, v) with label a, that is, $P_k = P_{k-1} + (u, v)$, with $P_k(u, v) = a$, then $i_k = a$.

We construct a sequence of nonnegative integers $I = \{i_0, i_1, i_2, ..., i_n\}$ as follows : let $i_0 = 0$ and $i_1, ..., i_n$ be the rearranged elements of S, U, $\hat{\pi}$ and $\check{\pi}$ in nondecreasing order. We have $n = |S| + |U| + |\pi|$.

Let $P_n = S$.
For k from n to 1 :
(a) if there is a cell $P_k(u, v) = i_k$, then erase this cell to obtain P_{k-1},
(b) else if the pair (i_k, x) belongs to π, then $P_{k-1} = ExtI(P_k, x)$,
(c)else if $U(u, v) = i_k$ and $P_k(u, v)$ exists (with label x), then $P_{k-1} = IntI(P_k, (u, v))$,
(d) else $P_{k-1} = EmpI(P_k, (u, v))$.
The tableaux P_k have respective shapes λ_k/μ_k. $P = (P_0, ..., P_n)$ and $I = \{i_0, i_1, ..., i_n\}$ satisfy that $i_k = a$ when $P_k = P_{k-1} + (u, v)$, with $P_k(u, v) = a$, so $I \in I(P)$.

Algorithm $\mathbf{OSCIL^{-1}}$.
(a) The input is (P, I) where $P \in \Theta_n(\emptyset_\alpha \to \beta/\mu)$ with $\mu \subseteq \alpha \cap \beta$ and $I \in I(P)$.
(b) The output is a triple $(S, U, \pi) \in \cup_{\mu \subseteq \alpha \cap \beta}[T(\beta/\mu) \times \overline{T}(\alpha/\mu)] \times GB$.

Let $\pi = \emptyset$, and $U_0 = P_0$.
For k from 1 to n:
(a)if $P_k = P_{k-1} + (u, v)$, then $U_i = U_{i-1}$,
(b) else $(P_k = P_{k-1} - (u, v))$, we have three cases :
(b_1) if the deletion is external (x ejected out of P_{k-1}), then add the pair (i_k, x) to π, $U_k = U_{k-1}$,

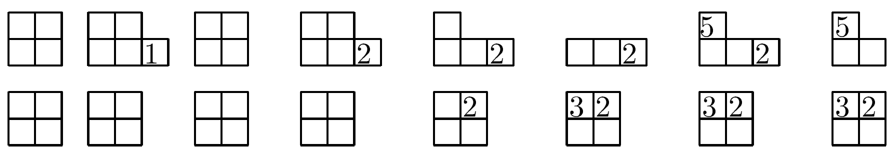
(b_2) else if it is internal (the cell $P_{k-1}(u, v)$ with label x is erased), then label the cell $U_{i-1}(u, v)$ with i_k to obtain U_k,

(b_3) else label the cell $U_{k-1}(u, v)$ with i_k to obtain U_k.
Finally, we obtain $S = P_n$, $U = U_n$ and $\pi \in GB$

Example 3.1.

k =	0	1	2	3	4	5	6	7
i_k =	0	1	1	2	2	3	5	6

$P_k =$, $U_k =$



$\pi_k =$ $\emptyset$ $\emptyset$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 6 & 1 \\ 2 & 1 \end{pmatrix}$

Theorem 1. *Let α, β be fixed partitions. There is a bijection Φ from triples (S, U, π) of*

$\cup_{\mu\subseteq\alpha\cap\beta}[T(\beta/\mu)\times\overline{T}(\alpha/\mu)]\times GB$ to (P,I) with a skew oscillating semi-standard tableau P of $\Theta_n(\emptyset_\alpha\rightarrow\beta/\mu)$, $n=|S|+|U|+|\pi|$ and $I=\{i_0,i_1,i_2,...,i_n\}\in I(P)$

Proof: For a triple $(S,U,\pi)\in\cup_{\mu\subseteq\alpha\cap\beta}[T(\beta/\mu)\overline{T}(\alpha/\mu)\times GB]$, we obtain directly a nondecreasing sequence $I=\{i_0,i_1,i_2,...,i_n\}$ with $i_0=0$ and $\{i_1,i_2,...,i_n\}$ rearranging the elements of S, U, $\hat{\pi}$ and $\check{\pi}$. A skew oscillating semi-standard tableaux $P\in\Theta_n$ results by applying the algorithm $OSCIL$.

To give the inverse operation, we construct a nondecreasing sequence $I=\{i_0,i_1,i_2,...,i_n\}$ from $P\in\Theta_n(\emptyset_\alpha\rightarrow\beta/\mu)$ as follows : (1) $i_0=0$ (2) if $P_k=P_{k-1}+(u,v)$, with $P_k(u,v)=x$ then $i_k=x$, else i_k is a positive integer satisfying $i_{k-1}\leq i_k\leq i_{k+1}$, so $I\in I(P)$. Next, we construct a sequence $(S_0,U_0,\pi_0)=(P_0,P_0,\emptyset)$, $(S_1,U_1,\pi_1),...,(S_n,U_n,\pi_n)=(S,U,\pi)$ from P and $I(P)$ by applying the algorithm $OSCIL^{-1}$. The algorithm $OSCIL^{-1}$ corresponds exactly to the inverse construction of the cell produced by the algorithm $OSCIL$. So (S,U,π) is in bijection with (P,I). Example 3.2 shows the application of the algorithm $OSCIL$ and $OSCIL^{-1}$. $\diamondsuit$

Remark 3.2 If the skew oscillating semi-standard tableaux $P\in\Theta_n(\emptyset_\alpha\rightarrow\beta/\alpha)$ has only insertion steps, the bijection Φ is $\Phi^{-1}(P,I)=(P_n,\emptyset_\alpha,\emptyset)$

$\overline{\Theta}_n(\beta/\mu\rightarrow\alpha/\gamma)$ denotes the set of skew oscillating tableaux of length n, $Q=(Q_0,Q_1,...,Q_n)$, satisfying the following conditions:

(1) the shape of Q_0 is β/μ, and the shape of Q_n is α/γ,

(2) Q_k is obtained from Q_{k-1} by erasing of a labelled cell (not by deletion algorithms) or an insertion of a cell by external insertion, internal insertion or empty insertion.

(3) if $x_i,x_j,...,x_m$ are deleted respectively from $Q_i,Q_j,...Q_m$, $i<j<...<m$, then $x_i\geq x_j\geq...\geq x_m$.

We know that $P=(P_0,P_1,...,P_n)\in\Theta_n(\alpha/\gamma\rightarrow\beta/\mu)$ if and only if $\overline{P}=(P_n,P_{n-1},...,P_0)\in\overline{\Theta}_n(\beta/\mu\rightarrow\alpha/\gamma)$.

We define a set of nonincreasing sequences of positive integers $\overline{I}(Q)=\cup_{j\in N}\overline{I}_j$,
$\overline{I}_j=\{\overline{j}_1,\overline{j}_2,...,\overline{j}_n\}$ for $Q\in\overline{\Theta}_n(\beta/\mu\rightarrow\alpha/\gamma)$ as follows:
(1) $\overline{j}_1\geq\overline{j}_2\geq...\geq\overline{j}_n$
(2) if $Q_{k+1}=Q_k-(u,v)$ with $Q_k(u,v)=x$, then $\overline{j}_k=x$.

Theorem 2. *Let π be a generalized biword of size $2n$ and α be an empty partition (of shape α). There is a bijection Φ_{RS} from pairs $(\emptyset_\alpha,\pi)$ to $\{(P,I_1),(Q,\overline{I}_2))\}$ of $\cup_\beta[\{\Theta_n(\emptyset_\alpha\rightarrow\beta/\alpha)\times I(P)\},\times\{\overline{\Theta}_n(\emptyset_\alpha\rightarrow\beta/\alpha)\times\overline{I}(Q)\}]$.*

Proof: According to the previous theorem, we obtain $(P_0=\emptyset_\alpha,...,P_n...,P_{2n}=\emptyset_\alpha)$
with $I=\{i_0,i_1,...i_n,..,i_{2n}\}$, and the result $(P,I_1)=(P_0,P_1,...,P_n(\text{of shape}\beta/\alpha),\{i_0,i_1,...,i_n\})$ with $I_1\in I(P)$, and $(Q,\overline{I}_2)=(P_{2n},P_{2n-1},...,P_n(\text{of shape}\beta/\alpha),\{i_{2n},i_{2(n-1)},...,i_n\})$. Therefore $Q\in\overline{\Theta}_n(\emptyset_\alpha\rightarrow\beta/\alpha)$ and $\overline{I}_2\in\overline{I}(Q)$ $\diamondsuit$

Taking an empty initial and final semi-standard tableaux in the theorem 1 and 2, we have an analog of Robinson-Schensted correspondence for oscillating semi-standard tableaux, as stated in the following results.

Corollaire 1. *Let β be fixed partitions and n a fixed integer. There is a bijection $\Phi_\emptyset$ from pairs (S,π) of $T(\beta)\times GB$ such that $n=|S|+|\pi|$ to pairs (P,I) with P of $\Theta_n(\emptyset\to\beta)$ and $I\in I(P)$.*

Corollaire 2. *Let n be a fixed integer. There is a bijection $\Phi_{RS_\emptyset}$ from generalized biwords π of GB_{2n} to pairs $\{(P,I_1),(Q,\overline{I}_2)\}$ of $\cup_\beta[(\Theta_n(\emptyset\to\beta)\times I(P))\times(\overline{\Theta}_n(\emptyset\to\beta)]\times\overline{I}(Q)]$.*

4. Geometric description of a generalized biword

In this section, we represent a geometric description of a generalized biword in the the first quadrant of the Cartesian plane by applying the geometric construction of Viennot in [1] and [13] for standard tableaux. We obtain an oscillating semi-standard tableau from the geometric description of
a generalized biword.

First, we present a method to standardize a generalized biword. Let $\mathbb{N}$ be a set of positive integers. For a given generalized biword $\pi=\begin{pmatrix} u_1 & u_2 & ... & u_n \\ v_1 & v_2 & ... & v_n\end{pmatrix}$, we propose a new alphabet $\mathbb{N}\cup\{j^{(h)}:j,h\in\mathbb{N}\}$ such that

$$...<j<j^{(1)}<j^{(2)}<...<j+1<j+1^{(1)}<j+1^{(2)}<...$$

If $u_j=u_{j+1}=...=u_{j+m}=v_{i_1}=v_{i_2}=...=v_{i_k}$, $i_1<i_2<...<i_k$, in π then we translate $v_{i_k}\to u_j$, $v_{i_{k-1}}\to u_j^{(1)}$, ..., $u_j\to u_j^{(m+k-1)}$.

For example,

$$\pi=\begin{pmatrix} 5 & 5 & 4 & 3 & 3 & 3 \\ 4 & 2 & 1 & 2 & 2 & 1\end{pmatrix}\leftrightarrow\tau=\begin{pmatrix} 5^{(1)} & 5 & 4^{(1)} & 3^{(2)} & 3^{(1)} & 3 \\ 4 & 2^{(2)} & 1^{(1)} & 2^{(1)} & 2 & 1\end{pmatrix}$$

The translation from π to τ is bijective and is called **standardization of** π. It is denoted by $\tau=\varphi(\pi)$. We know that τ has the properties of the generalized biword on the new alphabet $N\cup\{j^{(h)}:j,h\in N\}$.

Then, we represent $\tau=\varphi(\pi)$ in the place of π in the part $\{0,1,2,...,n\}\times\{0,1,...,n\}$ of the Cartesian plane as follows:

- Define a map $\Psi:$ abscissas x $(x=0,1,2,...,n)\to\{$ the greatest element of $\hat{\pi}+1\}\cup\hat{\tau}$ with

$$\Psi(x)=\begin{cases}\text{the greatest element of } \hat{\pi}+1 & \text{if } x=0\\ x^{th} \text{ greatest element of } \hat{\tau} & \text{else}\end{cases}$$

- Define a map $\Gamma:$ abscissas y $(y=0,1,2,...,n)\to\{0\}\cup\check{\tau}$ with

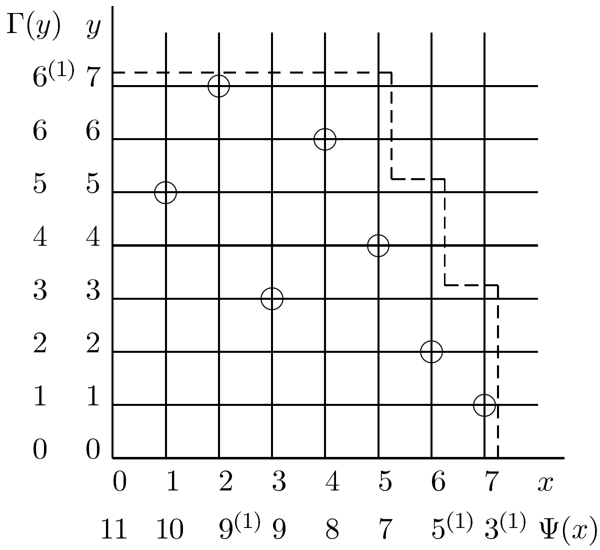
$$\Gamma(x)=\begin{cases}0 & \text{if } y=0\\ y^{th} \text{ lowest element of } \check{\tau} & \text{else}\end{cases}$$

- We define valid domain as the set of points (x,y) such that $\Psi(x)\geq\Gamma(y)$.
- For each pair (u_k,v_k) of τ, we set up the point $(\Psi^{-1}(u_k),\Gamma^{-1}(v_k))$ which is in the valid domain.

Example 4.1. For a generalized biword π of GB_{14} we obtain $\tau = \varphi(\pi)$ by the standardization of π :

$$\pi = \begin{pmatrix} 10 & 9 & 9 & 8 & 7 & 5 & 3 \\ 5 & 6 & 3 & 6 & 4 & 2 & 1 \end{pmatrix} \to \tau = \begin{pmatrix} 10 & 9^{(1)} & 9 & 8 & 7 & 5^{(1)} & 3^{(1)} \\ 5 & 6^{(1)} & 3 & 6 & 4 & 2 & 1 \end{pmatrix}.$$

Here we give the representation of τ. The dashed line describes the limit of valid domain, which is slightly extended on the figure for readability.



Description of $\tau = \varphi(\pi)$

Figure 4.1

From the Figure 4.1, we construct an oscillating semi-standard tableaux $T = \{T_0 = \emptyset, T_1, T_2, ..., T_{2n} = \emptyset\}$ as follows:

(a) let $A = \{\Psi(x_i)\}_{i=1..n}$ and $B = \{\Gamma(y_i)\}_{i=0..n}$. $I = \{i_0, ..., i_{2n}\}$ describes A and B lined up in nondecreasing order.

(b) For k from $2n$ to 1:
if $i_k \in A$ and $(\Psi^{-1}(i_k), y)$ is SW-corner of a Shadow line, then $T_{k-1} = ExtI(T_k, \Gamma(y))$.
else if $i_k \in B$ and $(x, \Gamma^{-1}(i_k))$ is SW-corner of a Shadow line, then $T_{k-1} = T_k - (u, v)$ with $T_k(u, v) = i_k$.

Figure 4.2 is the skew oscillating semi-standard tableaux corresponding to τ in Example 4.1. If $\Phi(\emptyset, \emptyset, \pi) = (P, J)$ in theorem 1, then we know that (P, J) is exactly equal to (T, I) if the exponents of contents of T and I are removed, resulting in the following theorem.

Theorem 3. *Let π be a generalized biword and $\tau = \varphi(\tau)$ and $D(\tau)$ be a description of τ with shadow lines in a Cartesian plane. Then there is a bijection from (T,I) of $\Theta_{2n}(\emptyset \to \emptyset) \times I(T)$*

to $D(\tau)$ of $\{D(\tau) | \tau = \varphi(\pi), \pi \in GB_{2n}\}$. If $\Phi(\emptyset, \emptyset, \pi) = (P, J)$, then (P, J) is equal to (T, I) in removing the exponents of contents of T and I.

Definition 1. *The shadow $S(\tau)$ of a generalized biword τ is the set of points (x, y) such that there is a point (x', y') of the representation of τ with $x' \leq x$, $y' \leq y$.*

Shadow lines of τ are defined recursively. The first shadow line L_1 of τ is the boundary of $S(\tau)$. To construct the shadow line L_{i+1} of τ remove the points of the representation of τ lying on L_i and construct the shadrow line of the remaining points. This procedure ends when there is no remaining point on the plane. The SW-coners of a shadow line are the points of the representation of τ located on this line ([8],[13]). The NE-coners of a shadow line are the points (x, y) of the shadow line such that $(x+1, y)$ and $(x, y+1)$ are not a part of this shadow line.

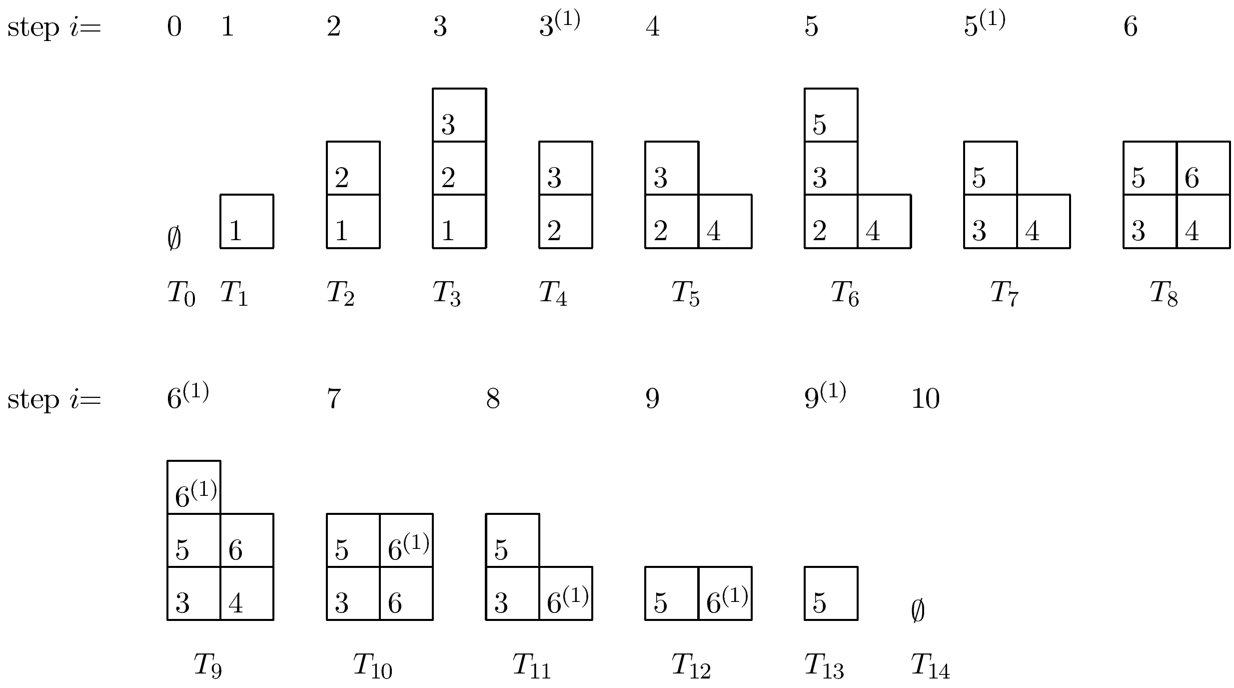
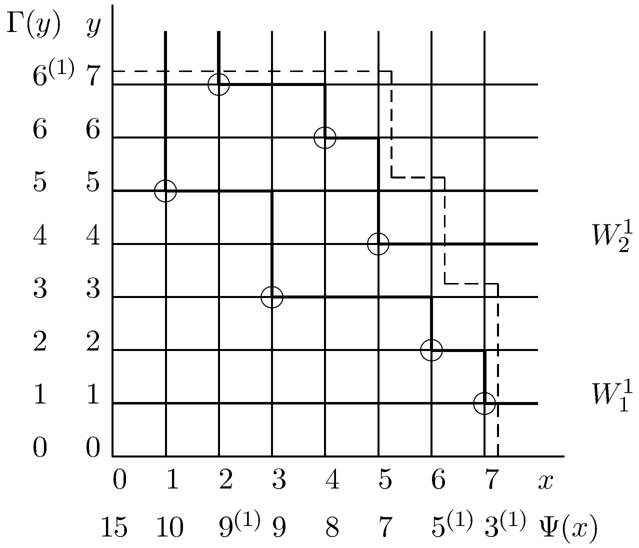


Figure 4.2

Definition 2. *The k^{th} skeleton of a generalized biword defined recursively by*
1. $\tau^{(1)} = \tau$
2. $\tau^{(k+1)} = \begin{pmatrix} \Psi(a_1) & \Psi(a_2) & \ldots & \Psi(a_m) \\ \Gamma(b_1) & \Gamma(b_2) & \ldots & \Gamma(b_n) \end{pmatrix}$ where $(a_1, b_1), \ldots, (a_m, b_m)$ are the NE-corners of $\tau^{(k)}$. The shadow diagram of τ is the set of shadow lines of all the skeletons $\tau^{(k)}$ of τ. The shadow lines of $\tau^{(k)}$ are denoted by $W_j^{(k)}$.

Example 4.2. Let π be the generalized biword of size $2n$ defined in example 4.1. Here we have the description of shadow lines $W_j^{(1)}$, $j = 1, 2$, of $\tau = \varphi(\pi)$.



Description of generalized biword $\tau = \varphi(\pi)$ with shadow lines

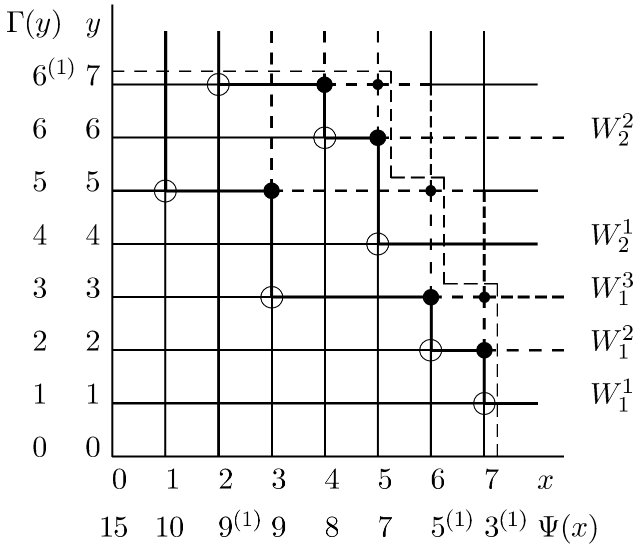
Figure 4.3

We can see that the shadow line W_1^1 in Figure 4.3 describes the behaviour of the first cell of the first row during the construction of $T_{14}, T_{13}, ..., T_0$. The shadow line W_1^1 has four SW-corners at (1,5), (3,3), (6,2) and (7,1). For the SW-corner (1,5), with $\Psi(1) = 10$ and $\Gamma(5) = 5$, followed by (3,3) with $\Psi(3) = 9$ and $\Gamma(3) = 3$. During the construction of the tableaux T_{14} to T_0, the first cell of first row is created during step 10 with label 5, this label is replaced during step 9 by the label 3. The label 3 is replaced during step $5^{(1)}$ by the label 2 and during step $3^{(1)}$ by the label 1, because $\Psi(6) = 5^{(1)}$ and $\Gamma(2) = 2$, $\Psi(7) = 3^{(1)}$ and $\Gamma(1) = 1$. The cell is deleted during step 1.

In the same way the shadow line W_j^i describes the behaviour of the j^{th} cell of the i^{th} row. So the theorem 4.1 in [1] is satisfied for a generalized biword and an oscillating semi-standard tableau as follows.

Theorem 4. *Let π be a generalized biword of size $2n$ and τ the standardization of π, i.e. $\tau = \varphi(\pi)$. If $\Phi(\emptyset, \emptyset, \tau) = (T, I)$ then the shadow line $W_j^{(i)}$ of τ describes the behavior of the j^{th} cell of the i^th row of the tableaux $T_{2n}, ..., T_0$ in the following way :*

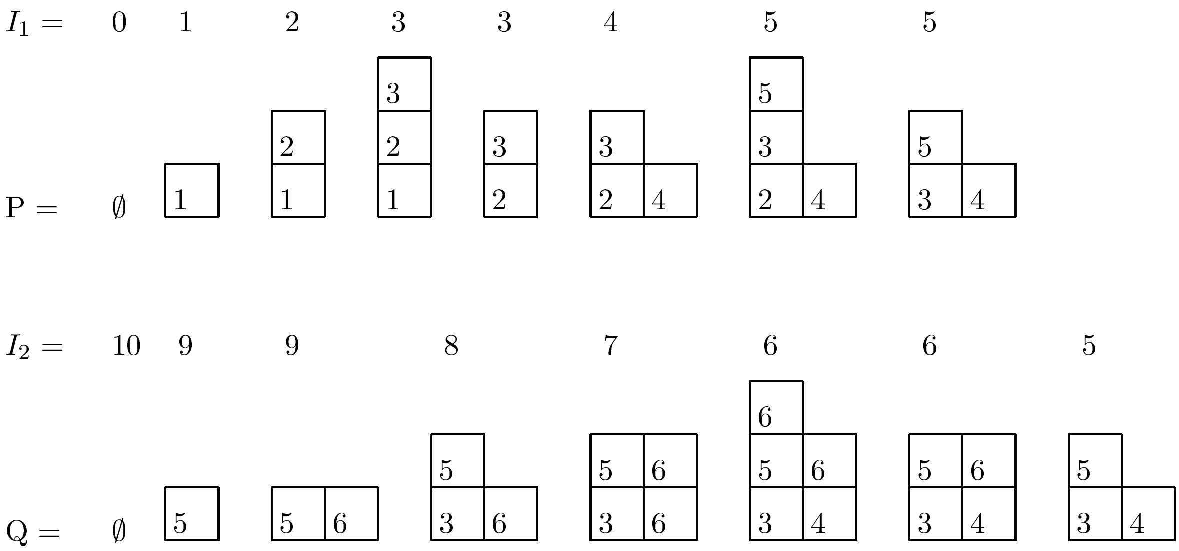
1. *a SW-corner (x, y) indicates that during step $\Psi(x)$, the label $\Gamma(y)$ fills in this cell,*
2. *when the line leaves the valid domain at (x, y), this cell is deleted during step $\Gamma(y)$,*
3. *otherwise, the cell remains unchanged.*



Description of a biword π

Figure 4.4

From Figure 4.2 the generalized biword π in Example 4.1 is in bijection with the following pairs $((P, I_1), (Q, I_2))$, where $(P, I_1) \in (\Theta_n(\emptyset \to \beta), I(P))$ and $(Q, I_2) \in (\overline{\Theta}_n(\emptyset \to \beta), \overline{I}(Q))$:



5. References

[1] C. CHAUVE and S. DULUCQ, *Properties of the Robinson-Schensted correspondence for oscillating and skew oscillating tableaux*, Technical Report RR-1211-99, LaBRI, Université Bordeaux I, 1999.

[2] S. H. CHOI and D. GOUYOU-BEAUCHAMPS, *Enumeration of generalized Young tableaux with bounded height*, Theoretical Computer Science, 117 (1993) 137-151.

[3] M. DELEST, S. Dulucq and L. FAVREAU, *An analogue to the Robinson-Schensted correspondence for oscillating tableaux* In Séminaire Lotharingien de Combinatoire

(Alghero,1988), Univ. Louis Pasteur, Strasbourg, 1988.

[4] S. Dulucq and B. E. *Sagan. La correspondance de Robinson-Schensted pour les tableaux oscillants gauches* Discrete Math., 139(1-3):129-142, 1995. Formal power series and algebraic combinatorics(Montreal, PQ, 1992).

[5] D. E. KNUTH, *Permutations, matrices and generalized Young tableaux*, Pacific J. Math., 34 (1970) 709-727.

[6] G. De B. ROBINSON, *On representation of the symmetric group*, Amer. J. Math. 60(1968), 745-760.

[7] B. E. SAGAN, *Ssifted tableaux, Schur Q-functions, and a conjecture of R. Stanley*, J. Combin. Theory Ser. A 45, (1987) 62-103.

[8] B. E. SAGAN, *The symmetric group: representations, combinatorial algorithms, and symmetric functions* Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991.

[9] B. E. SAGAN and R.P. STANLEY,*Robin-Schensted algorithms for Skew Tableaux*, J. Combin. Theory Ser. A 55, (1990) 161-193.

[10] C. SCHENSTED, *Longest increasing and decreasing subsequences*, Canad. J. Math., 13 (1961) 179-191.

[11] M.-P. Schützenberger, *La correspondance de Robinson*, In Combinatoire et representation du group symetrique, pages 59-113, Lecture Notes in Math., Vol. 579, berlin,1977.

[12] S. Sundaram. *The cauchy identity for Sp(2n)*, J. Combin. Theory Ser. A, 53(2): 209-238, 1990.

[13] G. Viennot, *Une forme géometrique de la correspondance de Robinson-Schensted*, In Combinatoire et representation du group symetrique, pages 29-58, Lecture Notes in Math., Vol. 579, Springer berlin,1977.

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