

# DESCENT NUMBER AND MAJOR INDICES FOR THE EVEN-SIGNED PERMUTATION GROUP

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ABSTRACT. We introduce and study three new statistics on the even-signed permutation group  $D_n$ . We show that two of these are Mahonian, i.e. are equidistributed with length, and that a pair of them gives a generalization of Carlitz's identity on the Euler-Mahonian distribution of the descent number and major index over  $S_n$ .

RÉSUMÉ. Nous présentons et étudions trois nouvelles statistiques sur le groupe de Coxeter de type  $D_n$ . Nous démontrons que parmi ces trois, deux sont "Mahonian", c'est-à-dire équidistribuées avec la longueur, et que deux autres portent à une généralisation de l'identité de Carlitz sur la distribution de Euler-Mahonian du nombre de descentes et du major index sur  $S_n$ .

## 1. INTRODUCTION

A well known classical result due to MacMahon (see [15]) asserts that the inversion number and the major index are equidistributed on the symmetric group. The joint distribution of major index and descent number was studied by Carlitz [7] and others. Several results of this nature have been generalized to the hyperoctahedral group  $B_n$  (see, e.g., [6],[13]) and many candidates for a major index for  $B_n$  have been suggested (see, e.g., [8],[9],[10],[12],[18]), but no generalizations of MacMahon's result have been found until the discovery of the flag major index in the recent paper [1]. After that, Foata posed the problem of finding a "descent statistic" that, together with the flag major index, allows the generalization to  $B_n$  of the well known Carlitz's identity on the Euler-Mahonian distribution of descent number and major index over  $S_n$ . In [2] Adin, Brenti and Roichman give two answers to Foata's question. Now it's natural to wonder if some of these statistics and results can be generalized to the even-signed permutation group  $D_n$ .

The goal of this paper is to show that this is the case. More precisely, we introduce and study three new statistics on  $D_n$ ; the  $D$ -negative descent number ( $ddes$ ), the  $D$ -negative major index ( $dmaj$ ) and the  $D$ -flag major index ( $fma j_D$ ). When restricted to  $S_n$ ,  $ddes$  reduces to descent number and  $dmaj$  to the major index. The two major indices on  $D_n$  are equidistributed with length, and the pair  $(ddes, dmaj)$  gives a generalization of Carlitz's identity to  $D_n$ .

The organization of this extended abstract is as follows. In the next section we collect some definitions, notation and results that are needed in the rest of the work. In §3 we introduce a new "descent set" and hence in a very natural way new definitions of "descent number" and "major index" on  $D_n$ . It's shown that  $dmaj$  is equidistributed with length and that  $(ddes, dmaj)$  gives a generalization of Carlitz's identity. In §4 we define, in terms of Coxeter elements, the  $D$ -flag major index for  $D_n$  and we show that it's equidistributed with length. Furthermore, we describe a combinatorial algorithm to compute it.

## 2. NOTATION, DEFINITIONS AND PRELIMINARIES

In this section we give some definitions, notation and results that will be used in the rest of this work. We let  $\mathbf{P} := \{1, 2, 3, \dots\}$ ,  $\mathbf{N} := \mathbf{P} \cup \{0\}$ , and  $\mathbf{Z}$  be the set of integers; for  $a \in \mathbf{N}$  we let  $[a] := \{1, 2, \dots, a\}$  (where  $[0] := \emptyset$ ). Given  $n, m \in \mathbf{Z}$ ,  $n \leq m$ , we let  $[n, m] := \{n, n+1, \dots, m\}$ . The cardinality of a set  $A$  will be denoted by  $|A|$  and we let  $\binom{[n]}{2} := \{S \subseteq [n] : |S| = 2\}$ . More generally, given a multiset  $M = \{1^{a_1}, 2^{a_2}, \dots, r^{a_r}\}$  we denote by  $|M|$  its cardinality, so  $|M| = \sum_{i=1}^r a_i$ . Given a variable  $q$  and a commutative ring  $R$  we denote by  $R[q]$  (respectively,  $R[[q]]$ ) the ring of polynomials (respectively, formal power series) in  $q$  with coefficient in  $R$ . For  $i \in \mathbf{N}$  we let, as customary,  $[i]_q := 1 + q + q^2 + \dots + q^{i-1}$  (so  $[0]_q = 0$ ).

Given a sequence  $\sigma = (a_1, \dots, a_n) \in \mathbf{Z}^n$  we say that  $(i, j) \in [n] \times [n]$  is an *inversion* of  $\sigma$  if  $i < j$  and  $a_i > a_j$ . We say that  $i \in [n-1]$  is a *descent* of  $\sigma$  if  $a_i > a_{i+1}$ . We denote by  $Inv(\sigma)$  and  $Des(\sigma)$  the set of inversions and the set of descents of  $\sigma$  and with  $inv(\sigma)$  and  $des(\sigma)$  their cardinality, respectively. We also let

$$(1) \quad maj(\sigma) := \sum_{i \in Des(\sigma)} i$$

and call it the *major index* of  $\sigma$ .

Let  $S_n$  be the set of all bijections  $\sigma : [n] \rightarrow [n]$ . If  $\sigma \in S_n$  then we write  $\sigma = \sigma_1 \dots \sigma_n$  to mean that  $\sigma(i) = \sigma_i$ , for  $i = 1, \dots, n$ . If  $\sigma \in S_n$  then we may also write  $\sigma$  in *disjoint cycle form* (see, e.g., [16, p.17]) and we will usually omit to write the 1-cycles of  $\sigma$ . For example, if  $\sigma = 64175823$  then we also write  $\sigma = (2, 4, 7)(1, 6, 8, 3)$ . Given  $\sigma, \tau \in S_n$  we let  $\sigma\tau := \sigma \circ \tau$  (composition of functions) so that, for example,  $(1, 2)(2, 3) = (1, 2, 3)$ .

We denote by  $B_n$  the group of all bijections  $\pi$  of the set  $[-n, n] \setminus \{0\}$  onto itself such that

$$\pi(-a) = -\pi(a)$$

for all  $a \in [-n, n] \setminus \{0\}$ , with composition as the group operation. This group is usually known as the group of *signed permutations* on  $[n]$ , or as the *hyperoctahedral group* of rank  $n$ . We identify  $S_n$  as a subgroup of  $B_n$ , and  $B_n$  as a subgroup of  $S_{2n}$ , in the natural ways.

If  $\pi \in B_n$  then we write  $\pi = [a_1, \dots, a_n]$  to mean that  $\pi(i) = a_i$  for  $i = 1, \dots, n$ , we call this the *window notation* of  $w$ , and we let

$$(2) \quad \begin{aligned} inv(\pi) &:= inv(a_1, \dots, a_n), \\ des(\pi) &:= des(a_1, \dots, a_n), \\ maj(\pi) &:= maj(a_1, \dots, a_n), \\ Neg(\pi) &:= \{i \in [n] : a_i < 0\}, \\ N_1(\pi) &:= |Neg(\pi)|, \end{aligned}$$

and

$$(3) \quad N_2(\pi) := |\{\{i, j\} \in \binom{[n]}{2} : a_i + a_j < 0\}|.$$

We denote by  $D_n$  the subgroup of  $B_n$  consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

$$D_n := \{\pi \in B_n : N_1(\pi) \equiv 0 \pmod{2}\}.$$

Obviously, the definitions in (2) and (3) are still valid for  $\pi \in D_n$ .

It is well known (see, e.g., [5, §8.2]) that  $D_n$  is a Coxeter group with respect to the generating set  $S := \{s_0, s_1, \dots, s_{n-1}\}$  where

$$s_0 := [-2, -1, 3, \dots, n]$$

and

$$s_i := [1, 2, \dots, i-1, i+1, i, i+2, \dots, n]$$

for  $i = 1, \dots, n-1$ . This gives rise to another natural statistic on  $D_n$  the *length* (similarly definable for any Coxeter group), namely

$$l(\pi) := \min\{r \in \mathbf{N} : \pi = s_{i_1} \dots s_{i_r} \text{ for some } i_1, \dots, i_r \in [0, n-1]\}.$$

There is a well known direct combinatorial way to compute this statistic for  $\pi \in D_n$  (see, e.g., [5, §8.2]), namely

$$(4) \quad l(\pi) = \text{inv}(\pi) - \sum_{i \in \text{Neg}(\pi)} \pi(i) - N_1(\pi).$$

It's not hard to prove that for all  $\pi \in B_n$  (and so also for  $\pi \in D_n$ ),

$$(5) \quad N_1(\pi) + N_2(\pi) = - \sum_{i \in \text{Neg}(\pi)} \pi(i),$$

so equivalently (4) becomes

$$(6) \quad l(\pi) = \text{inv}(\pi) + N_2(\pi).$$

For example, if  $\pi := [-4, 1, 3, -5, -2, -6] \in D_6$  then  $\text{inv}(\pi) = 10$ ,  $\text{des}(\pi) = 2$ ,  $\text{maj}(\pi) = 8$ ,  $N_1(\pi) = 4$ ,  $N_2(\pi) = 13$  and  $l(\pi) = 23$ .

We follow [5] for general Coxeter group notation and terminology. In particular, let  $(W, S)$  be a Coxeter system, for  $J \subseteq S$  we let  $W_J$  be the subgroup of  $W$  generated by  $J$ , and

$$W^J := \{w \in W : l(ws) > l(w) \text{ for all } s \in J\}.$$

We call  $W_J$  the *parabolic subgroup* generated by  $J$  and  $W^J$  the *set of minimal left coset representatives* of  $W_J$  or the *quotient*. The quotient  $W^J$  is a poset according to the Bruhat order (see, e.g., [5] or [14]).

The following is well known, (see, e.g., [5] or [14]).

**Proposition 1.** *Let  $J \subseteq S$ . Then:*

- i) *Every  $w \in W$  has a unique factorization  $w = w^J w_J$  such that  $w^J \in W^J$  and  $w_J \in W_J$ .*
- ii) *For this factorization  $l(w) = l(w^J) + l(w_J)$ .*

Now we let

$$(7) \quad T := \{\pi \in D_n : \text{des}(\pi) = 0\}.$$

It is well known, and easy to see, that

$$(8) \quad D_n = \bigsqcup_{u \in S_n} \{\pi u : \pi \in T\},$$

where  $\bigsqcup$  denotes disjoint union. We will often use this decomposition in this paper. Note that (8) is one case of the multiplicative decomposition of a Coxeter group into a parabolic subgroup and its minimal coset representatives (see Proposition 1), more precisely  $T$  is the quotient corresponding to the maximal parabolic subgroup generated by  $J := S \setminus \{s_0\}$ . In the next section we will analyze this issue in more detail.

For  $n \in \mathbf{P}$  we let

$$A_n(t, q) := \sum_{\sigma \in S_n} t^{\text{des}(\sigma)} q^{\text{maj}(\sigma)},$$

and  $A_0(t, q) := 1$ . For example,  $A_3(t, q) = 1 + 2tq^2 + 2tq + t^2q^3$ . The following result is due to Carlitz, and we refer the reader to [7] for its proof.

**Theorem 2.** *Let  $n \in \mathbf{P}$ . Then*

$$(9) \quad \sum_{r \geq 0} [r + 1]_q^{n_t r} = \frac{A_n(t, q)}{\prod_{i=0}^{n-1} (1 - tq^i)}$$

in  $\mathbf{Z}[q][[t]]$ .

### 3. THE “NEGATIVE” STATISTICS

In this section we define a new “descent set” for elements of  $D_n$ . This gives rise, in a very natural way, to the definitions of “major index” and “descent number” for  $D_n$ . We then show that these two statistics give a generalization of Carlitz’s identity to  $D_n$ , and that the former is equidistributed with length.

**3.1. The  $D$ -Negative Descent Multiset.** For  $\pi \in D_n$  let

$$Des(\pi) := \{i \in [n - 1] : \pi(i) > \pi(i + 1)\},$$

we define the *D-negative descent multiset*

$$(10) \quad DDes(\pi) := Des(\pi) \uplus \{-\pi(i) - 1 : i \in Neg(\pi)\} \setminus \{0\}.$$

For example, if  $\pi = [-4, 1, 3, -5, -2, -6] \in D_6$  then  $Des(\pi) = \{3, 5\}$  and  $DDes(\pi) = \{1, 3^2, 4, 5^2\}$ .

Note that if  $\pi \in S_n$  then  $DDes(\pi)$  is a set and coincides with the usual descent set of  $\pi$ . Also, note that  $DDes(\pi)$  can be defined rather naturally also in purely Coxeter group theoretic terms. In fact, for  $i \in [n - 1]$  let  $\xi_i \in D_n$  be defined by

$$\xi_i := [-1, 2, \dots, i, -i - 1, i + 2, \dots, n].$$

Then  $\xi_1, \dots, \xi_{n-1}$  are reflections (in the Coxeter group sense, see e.g., [5] or [14]) of  $D_n$  and it is clear from (4) that

$$DDes(\pi) := \{i \in [n - 1] : l(\pi s_i) < l(\pi)\} \uplus \{i \in [n - 1] : l(\pi^{-1} \xi_i) < l(\pi^{-1})\}.$$

These considerations explain why it is natural to think of  $DDes(\pi)$  as a “descent set”, so the following definitions are natural.

For  $\pi \in D_n$  we let

$$dDes(\pi) := |DDes(\pi)|$$

and

$$dmaj(\pi) := \sum_{i \in DDes(\pi)} i.$$

For example if  $\pi = [-4, 1, 3, -5, -2, -6] \in D_6$  then  $dDes(\pi) = 6$ , and  $dmaj(\pi) = 21$ .

Note that from (10) there follows that

$$(11) \quad dmaj(\pi) = maj(\pi) - \sum_{i \in Neg(\pi)} \pi(i) - N_1(\pi) = maj(\pi) + N_2(\pi).$$

This formula is also one the motivations behind our definition of  $dmaj(\pi)$ , because of the corresponding formulas (4) and (6), (see also [2]) .

Also note that

$$(12) \quad dDes(\pi) = des(\pi) + N_1(\pi) + \epsilon(\pi),$$

where

$$\epsilon(\pi) := \begin{cases} -1 & \text{if } 1 \notin \pi([n]) \\ 0 & \text{if } 1 \in \pi([n]). \end{cases}$$

**3.2. Equidistribution.** Our first result shows that  $d\text{maj}$  and  $l$  are equidistributed in  $D_n$ .

**Proposition 3.** [4, Proposition 3.1] *Let  $n \in \mathbf{P}$ . Then*

$$\sum_{\pi \in D_n} q^{d\text{maj}(\pi)} = \sum_{\pi \in D_n} q^{l(\pi)}.$$

**3.3. Generalization of Carlitz's Identity.** We start with some notation and terminology concerning partitions (see [17, §7.2]). By an (integer) *strict partition* we mean a sequence of positive integers  $\lambda = (\lambda_1, \dots, \lambda_k)$  such that  $\lambda_1 > \lambda_2 > \dots > \lambda_k$ . We denote by  $|\lambda| := \sum_i \lambda_i$ . We denote by  $\tilde{P}_S$  the set of all (integer) strict partitions. Given  $n \in \mathbf{P}$  we let

$$\tilde{P}_S(n) := \{\lambda \in \tilde{P}_S : \lambda \subseteq (n, n-1, \dots, 2, 1)\}.$$

As before, let  $T = \{\pi \in D_n : \text{des}(\pi) = 0\}$  so

$$T = \{\pi \in D_n : \pi(1) < \pi(2) < \dots < \pi(n)\}.$$

Therefore, given  $\pi \in T, \pi \neq e$ , there is a unique  $k \in [n]$  such that

$$\pi(k) < 0 < \pi(k+1).$$

Given  $\pi \in T$  we associate to  $\pi$  the strict partition

$$(13) \quad \Lambda(\pi) := (-\pi(1) - 1, -\pi(2) - 1, \dots, -\pi(k) - 1).$$

The following is known, (see, e.g., [3]).

**Proposition 4.** *The map  $\Lambda$  defined by (13) is a bijection between  $T$  and  $\tilde{P}_S(n-1)$ . Furthermore  $\pi \leq \sigma$  in  $T$  if and only if  $\Lambda(\pi) \subseteq \Lambda(\sigma)$  and  $l(\pi) = |\Lambda(\pi)|$  for all  $\pi, \sigma \in T$ .*

We begin with the following lemma.

**Lemma 1.** [4, Lemma 3.3] *Let  $n \in \mathbf{P}$ . Then*

$$\sum_{\sigma \in T} t^{N_1(\sigma) + \epsilon(\sigma)} q^{N_2(\sigma)} = \sum_{S \subseteq [n-1]} t^{|S|} q^{\sum_{i \in S} i} = \prod_{i=1}^{n-1} (1 + tq^i).$$

We are now ready to state the main result of this work, namely that the pair of statistics  $(d\text{des}, d\text{maj})$  solves Foata's problem for the group of even-signed permutations  $D_n$ .

**Theorem 5.** [4, Theorem 3.5] *Let  $n \in \mathbf{P}$ . Then*

$$(14) \quad \sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\pi \in D_n} t^{d\text{des}(\pi)} q^{d\text{maj}(\pi)}}{(1-t)(1-tq^n) \prod_{i=1}^{n-1} (1-t^2q^{2i})}$$

in  $\mathbf{Z}[q][[t]]$ .

Note that, as in (9) for  $S_n$  and in ([2, Theorem 3.2]) for  $B_n$ , the powers of  $q$  in the denominator of formula (14) are the Coxeter degrees of  $D_n$  (see [14, p.59]).

#### 4. THE FLAG MAJOR INDEX FOR $D_n$

In this section we introduce another new "major index" statistic for  $D_n$ . This is an analogue of the flag major index introduced in [1]. We show that this statistic is equidistributed with length and we give a combinatorial algorithm to compute it.

**4.1. The  $D$ -Flag Major Index.** For  $i = 0, \dots, n-1$  we define

$$(15) \quad t_i := s_i s_{i-1} \cdots s_0,$$

explicitly for all  $i \in [n-1]$

$$(16) \quad t_i = [-1, -i-1, 2, 3, \dots, i, i+2, \dots, n],$$

and for  $i = 0$

$$(17) \quad t_0 = [-2, -1, 3, \dots, n] = s_0.$$

These are Coxeter elements (see e.g., [14, §3.16]), in a distinguished flag of parabolic subgroups

$$1 < G_1 < G_2 < \dots < G_n = D_n$$

where  $G_i \simeq D_i$  ( $i \geq 2$ ) is the parabolic subgroup of  $D_n$  generated by  $s_0, s_1, \dots, s_{i-1}$ . The family  $\{t_i\}_i$  is a new set of generators for  $D_n$ , and we have the following proposition.

**Proposition 6.** [4, Proposition 4.1] *For every  $\pi \in D_n$  there exists a unique representation*

$$(18) \quad \pi = t_0^{h_{n-1}} t_{n-1}^{k_{n-1}} t_0^{h_{n-2}} t_{n-2}^{k_{n-2}} \cdots t_0^{h_1} t_1^{k_1}$$

with  $0 \leq h_r \leq 1$ ,  $0 \leq k_r \leq 2r-1$  and

$$(19) \quad k_r \in \{2r-1, r-1\} \text{ if } h_r = 1$$

for all  $r = 1, \dots, n-1$ .

Note that the representation (18) is not unique if we drop the condition (19). For example consider  $\pi = [3, -2, 1, -4] \in D_4$ , then  $\pi$  has two different representations of type (18), namely,  $\pi = t_3^2 t_0^2 t_2^2 t_0$  and  $\pi = t_0 t_3^2 t_2 t_0 t_1$ . The representation of Proposition 6 is the first one.

Let  $\pi \in D_n$ , then we define the  *$D$ -flag major index* of  $\pi$  by

$$(20) \quad fma_j_D := \sum_{i=1}^{n-1} k_i + \sum_{i=1}^{n-1} h_i.$$

**4.2. Equidistribution.** For  $0 \leq m \leq 2n-1$ ,  $n \geq 2$ , we define  $r_{n,m} \in D_n$  as follows: for  $n = 2$ ,

$$r_{2,m} := \begin{cases} e & \text{if } m = 0 \\ s_1 & \text{if } m = 1 \\ s_1 s_0 & \text{if } m = 2 \\ s_0 & \text{if } m = 3, \end{cases}$$

and for  $n > 2$ ,

$$r_{n,m} := \begin{cases} e & \text{if } m = 0 \\ s_{n-m} s_{n-m+1} \cdots s_{n-1} & \text{if } 0 < m < n \\ s_{m-n+1} s_{m-n} \cdots s_0 s_2 s_3 \cdots s_{n-1} & \text{if } n \leq m < 2n-1 \\ s_0 s_2 s_3 \cdots s_{n-1} & \text{if } m = 2n-1. \end{cases}$$

The set  $\{r_{n,m} : 0 \leq m < 2n\}$  forms a complete set of representatives of minimal length for the left cosets of  $D_{n-1}$  in  $D_n$ . Moreover this is still valid for every  $i \in [2, n]$ , namely,  $r_{i,m} \in D_i^{J_i}$  for all  $m \in [0, 2i-1]$ , where  $J_i := S \setminus \{s_{n-1}, \dots, s_{i-1}\}$ . Hence we have the following decomposition

$$D_n = D_n^{J_n} D_{n-1}^{J_{n-1}} \cdots D_2.$$

Note that the length of  $r_{i,m}$  is  $\bar{m}_i$ , where

$$\bar{m} := \begin{cases} m & \text{if } 0 \leq m \leq 2i - 2 \\ i - 1 & \text{if } m = 2i - 1. \end{cases}$$

From *i*) of Proposition 1 we know that each element  $\pi \in D_n$  has a unique representation as a product

$$(21) \quad \pi = \prod_{k=1}^{n-1} r_{n+1-k, m_{n+1-k}}$$

where  $0 \leq m_j < 2j$  for all  $j$ . From *ii*) of Proposition 1 there follows that

$$(22) \quad l(\pi) = \sum_{j=2}^n \bar{m}_j.$$

Thanks to the unique representation (21) we define a map  $\phi : D_n \rightarrow D_n$  in the following way:

$$\phi\left(\prod_{k=1}^{n-1} r_{n+1-k, m_{n+1-k}}\right) := \prod_{k=1}^{n-1} \phi(r_{n+1-k, m_{n+1-k}}),$$

where for  $i \neq 2$ ,

$$\phi(r_{i,m}) := \begin{cases} t_{i-1}^m & \text{if } m < 2i - 2 \\ t_0 t_{i-1}^{\bar{m}-1} & \text{if } 2i - 2 \leq m \leq 2i - 1, \end{cases}$$

and for  $i = 2$ ,

$$\phi(r_{2,m}) := \begin{cases} e & \text{if } m = 0 \\ t_1 & \text{if } m = 1 \\ t_0 t_1 & \text{if } m = 2 \\ t_0 & \text{if } m = 3. \end{cases}$$

The definition of  $\phi$ , together with Proposition 6 and (21), imply the following result.

**Proposition 7.** [4, Proposition 4.2] *The map  $\phi : D_n \rightarrow D_n$  is a bijection.*

This implies the main result of this section, namely that the  $D$ -flag major index is equidistributed with the length in  $D_n$ .

**Theorem 8.** [4, Theorem 4.3] *Let  $n \in \mathbf{P}$ . Then*

$$\sum_{\pi \in D_n} q^{f\text{maj}_D(\pi)} = \sum_{\pi \in D_n} q^{l(\pi)}.$$

Note that the  $B$ -flag major index ( $f\text{maj}$ ) defined on  $B_n$  (see [1]) does not work on  $D_n$ . Namely if we consider  $\pi \in D_n$  as an element of  $B_n$ , then  $f\text{maj}(\pi)$  is not equidistributed with length on  $D_n$ . For example let  $\pi = [-2, -1]$  then  $f\text{maj}_D(\pi) = 1$  while  $f\text{maj}(\pi) = 4$ , and in  $D_2$  there is no element of length 4.

Note also that  $f\text{maj}_D$  restricted to  $S_n$  is not the major index and it's not equidistributed with length. It seems to be a new statistic on  $S_n$ . It's easy to see that for each  $\pi \in S_n$ ,  $f\text{maj}_D(\pi)$  is always even and that  $f\text{maj}_D(\pi) \geq \text{maj}(\pi)$ . If we let  $E_n(q) := \sum_{\pi \in S_n} q^{f\text{maj}_D(\pi)}$ , for  $n \leq 4$  we have  $E_1(q) = 1$ ,  $E_2(q) = 1 + q^2$ ,  $E_3(q) = 1 + 3q^2 + q^4 + q^6$  and  $E_4(q) = 1 + 5q^2 + 6q^4 + 7q^6 + 3q^8 + q^{10} + q^{12}$ .

**4.3. A combinatorial algorithm.** Let  $\sigma = (a_1, \dots, a_n) \in \mathbf{Z}^n$ , we use this *split-notation*

$$\sigma = [a_1][a_2, \dots, a_{i+1}][a_{i+2}, \dots, a_n].$$

Sometimes it will be useful to denote the first part with  $A$  and the second with  $C_i$  where  $i$  represents the number of its elements.

We define the following operations on  $\sigma \in \mathbf{Z}^n$ :

$$\overset{-0}{\sigma}_i := [-a_2][-a_1, a_3, \dots, a_{i+1}][a_{i+2}, \dots, a_n],$$

and

$$\overset{-1}{\sigma}_i := [-a_1][-a_{i+1}, a_2, \dots, a_i][a_{i+2}, \dots, a_n].$$

In these cases we will write  $\overset{-0}{\sigma}_i = (A^0, C_i^0, [a_{i+2}, \dots, a_n])$  and  $\overset{-1}{\sigma}_i = (\overset{-1}{A}, \overset{-1}{C}_i, [a_{i+2}, \dots, a_n])$ . Moreover for all  $n \in \mathbf{P}$  we define

$$(23) \quad \rightarrow_i^n := \rightarrow_i^1 \circ \dots \circ \rightarrow_i^1 \quad n\text{-times.}$$

Note that for every  $\sigma \in \mathbf{Z}^n$ ,  $\overset{-2i}{\sigma}_i = \sigma$ .

For example let  $\pi \in D_4$ ,  $\pi = [-2][1, 3, -4, 5] = (A, C_4)$ , then

$$\overset{-0}{\pi}_4 = [-1][2, 3, -4, 5] = (A^0, C_4^0),$$

$$\overset{-5}{\pi}_4 = [2][5, -1, -3, 4] = (\overset{-5}{A}, \overset{-5}{C}_4),$$

and

$$\overset{-2}{\pi}_3 = [-2][-3, 4, 1][5] = (\overset{-2}{A}, \overset{-2}{C}_3, [5]).$$

These are the two technical properties that we will use in the algorithm. Fix  $i \in [n-1]$ , let  $t_i$  be as in (16),

$$t_i = [-1][-i-1, 2, 3, \dots, i][i+2, \dots, n].$$

It's easy to see that for all  $i \in [n-1]$  we have

$$(24) \quad t_i^2 = t_i t_i = \overset{-1}{t}_i,$$

and by (23) that for  $k \in \mathbf{P}$

$$(25) \quad t_i^k = \overset{-k-1}{t}_i.$$

Now consider  $t_{i-1} = [-1][-i, 2, \dots, i-1][i+1, \dots, n]$ . As before it is not hard to see that

$$(26) \quad t_i t_{i-1} = \overset{-1}{t}_{i-1}.$$

Now we are able to state the algorithm to compute the unique representation of  $\pi$  as in Proposition 6, namely

$$\pi = f_{n-1} \cdots f_1$$

where for all  $r \in [n-1]$ ,  $f_r = t_0^{h_r} t_r^{k_r}$  with  $h_r \in [0, 1]$  and  $k_r \in [0, 2r-1]$ .

We construct a sequence  $e_0, \dots, e_{n-1}$  of elements of  $D_n$  such that

- i)  $e_0 = e$ ,  $e_{n-1} = \pi$ ;
- ii)  $e_i = f_{n-1} \cdots f_{n-i}$ , for all  $i \in [1, n-1]$ ;
- iii)  $\pi(j) = e_i(j)$ , for all  $j > n-i$ .

From *iii*) there immediately follows that  $e_{n-1} = \pi$ .

We need to do  $n - 1$  steps. From now to avoid confusion we put on  $A$  an index corresponding to the number of steps. We begin with  $e_0 = [1][2, \dots, n]$ . We describe the  $(n - i + 1)$ -th step. Assume that  $e_{n-i}$  has been constructed. Then by *iii*),

$$e_{n-i} = (A_{n-i}, C_{i-1}, [\pi(i+1), \dots, \pi(n)]).$$

For simplicity, we define  $p(i)$  and  $p(-i)$  to be the positions of  $\pi(i)$  and  $-\pi(i)$  in  $C_{i-1}$  or  $C_{i-1}^0$  respectively. There are four cases to consider.

1)  $\pi(i) \in C_{i-1}$

Then we let  $k_{i-1} = i - 1 - p(i)$  and  $h_{i-1} = 0$ . Hence  $f_{i-1} = t_{i-1}^{i-1-p(i)}$ .

2)  $-\pi(i) \in C_{i-1}$

Then we let  $k_{i-1} = 2i - 2 - p(-i)$  and  $h_{i-1} = 0$ . Hence  $f_{i-1} = t_{i-1}^{2i-2-p(-i)}$ .

3)  $\pi(i) \in A_{n-i}$

Then  $-\pi(i) \in C_{i-1}^0$  and in particular  $p(-i) = 1$ . We let  $k_{i-1} = 2i - 3$  and  $h_{i-1} = 1$ . Hence  $f_{i-1} = t_0 t_{i-1}^{2i-3}$ .

4)  $-\pi(i) \in A_{n-i}$

Then  $\pi(i) \in C_{i-1}^0$  and  $p(i) = 1$ . We let  $k_{i-1} = i - 2$  and  $h_{i-1} = 1$ . Hence  $f_{i-1} = t_0 t_{i-1}^{i-2}$ .

We have determined the factor  $f_{i-1}$ . From (23) and (25) there follows that  $e_{n-i+1}(i) = \pi(i)$  and by (26) *iii*) again holds.

Therefore

$$e_{n-i+1} = (A_{n-i+1}, C_{i-2}, [\pi(i), \dots, \pi(n)]),$$

where in cases 1) and 2),

$$A_{n-i+1} := A_{n-i}^{\rightarrow k_{i-1}}, \quad C_{i-2} := C_{i-1}^{\rightarrow k_{i-1}} \setminus [\pi(i)],$$

while in cases 3) and 4),

$$A_{n-i+1} := A_{n-i}^0, \quad C_{i-2} := C_{i-1}^0 \setminus [\pi(i)].$$

Observe that in the first step  $p(n) = \pi(n) - 1$  and  $p(-n) = -\pi(n) + 1$ . These can be used for the computation of  $e_1$ .

We finish this section by illustrating the procedure with an example.

Let  $\pi = [5, 3, -4, 1, -2] \in D_5$ . We start from

$$e = e_0 = [1][2, 3, 4, 5] = (A_0, C_4).$$

1<sup>st</sup> - step)  $-\pi(5) = -2 \in C_4$

We are in case 2) and  $p(-5) = 1$ , so  $k_4 = 7$ ,  $h_4 = 0$  and  $f_4 = t_4^7$ .

It follows that  $A_1 = A_0^{\rightarrow 7} = [-1]$  and  $C_3 = C_4^{\rightarrow 7} \setminus [-2] = [3, 4, 5]$ . Hence,

$$e_1 = [-1][3, 4, 5] [-2].$$

2<sup>nd</sup> - step)  $-\pi(4) = -1 \in A_1$

We are in case 4) so  $k_3 = 2$ ,  $h_3 = 1$  and  $f_3 = t_0 t_3^2$ .

It follows that  $A_2 = A_1^{\rightarrow 2} = [-3]$  and  $C_2 = C_3^{\rightarrow 2} \setminus [1] = [-4, -5]$ . Hence,

$$e_2 = [-3] [-4, -5] [1, -2].$$

3<sup>rd</sup> - step)  $\pi(3) = -4 \in C_2$

We are in case 1) and  $p(3) = 1$ , so  $k_2 = 1$ ,  $h_2 = 0$  and  $f_2 = t_2$ .

It follows that  $A_3 = \overset{\rightarrow 1}{A_2} = [3]$  and  $C_1 = \overset{\rightarrow 1}{C_2} \setminus [-4] = [5]$ . Hence,

$$e_3 = [5][-4, 1, -2].$$

4<sup>th</sup> – step)  $\pi(2) = 3 \in A_3$

We are in case 3) so  $k_1 = 1$ ,  $h_1 = 1$  and  $f_1 = t_0 t_1$ .

It follows that  $A_4 = \overset{\rightarrow 1}{A_3} = [5]$  and  $C_0 = \emptyset$ . Hence,

$$e_4 = [5][3, -4, 1, -2] = \pi,$$

and we are done. Finally  $\pi = t_4^7 t_0 t_3^2 t_2 t_0 t_1$  and  $f_{maj_D}(\pi) = 12$ .

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