LITTLEWOOD-RICHARDSON COEFFICIENTS AND HOOK INTERPOLATIONS (EXTENDED ABSTRACT)

RON M. ADIN, AVITAL FRUMKIN, AND YUVAL ROICHMAN

Abstract. The hook components of $V^{\otimes n}$ interpolate between the symmetric power $\mathrm{Sym}^n(V)$ and the exterior power $\wedge^n(V)$. When V is the vector space of $k \times m$ matrices, a decomposition of the hook components into irreducibles involving convolutions of Littlewood-Richardson coefficients is presented. Classical theorems of Ehresmann, Thrall, Helgason, James, Shimura and others are proved as boundary cases.

Résumé. Les composants d'équerres de $V^{\otimes den}$ interpolent entre la puissance symétrique $\mathrm{Sym}^n(v)$ et la puissance extérieure $\wedge^n(v)$. Quand V est l'espace vectoriel des matrices $k \times m$, une décomposition des composantes d'équerres en composantes irréductibles comprenant des convolutions de coefficients de Littlewood-Richardson est présentée. Des théorèmes classiques d'Ehresmann, de Thrall, de Helgason, de James, de Shimura et de d'autres sont prouvés comme des cas limites.

1. Introduction

The vector space $M_{k,m}$ of $k \times m$ matrices over $\mathbb{C}$ carries a (left) $GL_k(\mathbb{C})$-action and a (right) $GL_m(\mathbb{C})$-action. A classical Theorem of Ehresmann [3] describes the decomposition of an exterior power of $M_{k,m}$ into irreducible bimodules. The symmetric analogue was given later (cf. [7]). See Section 4 below.

In this paper we present a natural interpolation between these theorems, in terms of hook components of the n-th tensor power of $M_{k,m}$. This interpolation involves convolutions of the Littlewood-Richardson coefficients. Duality and asymptotics of the decomposition of hook components follow.

Similar concepts are applied to the diagonal two-sided $GL_k(\mathbb{C})$-action on the vector space of $k \times k$ matrices. Classical theorems of Thrall [19] and James [8] (for the symmetric powers of symmetric matrices), and of Helgason [5], Shimura [15] and Howe [6] (for the symmetric powers of anti-symmetric matrices) are extended, and a bivariate interpolation is presented. This interpolation involves natural extensions of the Littlewood-Richardson coefficients.

Proofs are obtained using the representation theory of the symmetric and hyperoctahedral groups, together with plethysm of symmetric functions and Schur-Weyl duality. The techniques are different in spirit from those used in the classical works cited above, except for [8].

The interpolations presented have surprising combinatorial implications, which will be studied elsewhere.

2. Definitions and Notations

Let n be a positive integer. A *partition* of n is a vector of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\lambda_1 + \ldots + \lambda_k = n$. We denote this by

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Research was supported in part by the Israel Science Foundation, founded by the Israel Academy of Sciences and Humanities, and by internal research grants from Bar-Ilan University.

$\lambda \vdash n$. The *size* of a partition $\lambda \vdash n$, denoted $|\lambda|$, is n, and its *length*, $\ell(\lambda)$, is the number of parts. The empty partition $\emptyset$ has size and length zero: $|\emptyset| = \ell(\emptyset) = 0$. The set of all partitions of n with at most k parts is denoted by $\mathrm{Par}_k(n)$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ define the *conjugate partition* $\lambda' = (\lambda'_1, \ldots, \lambda'_t)$ by letting λ'_i be the number of parts of λ that have size at least i.

A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ may be viewed as the subset

$$\{(i,j) \mid 1 \le i \le k, 1 \le j \le \lambda_i\} \subseteq \mathbb{Z}^2,$$

the corresponding *Young diagram*. Using this interpretation, we may speak of inclusion $\mu \subseteq \lambda$, intersection $\lambda \cap \mu$ and the set difference $\lambda \setminus \mu$ of any two partitions. The set difference is called a *skew shape*; when $\mu \subseteq \lambda$ it is usually denoted λ/μ.

A *semistandard Young tableau* of shape λ/μ is obtained by inserting positive integers as entries in the cells of the Young diagram of shape λ, so that the entries weakly increase along rows and strictly increase down columns. The *content vector* of a semistandard Young tableau T $\mathit{cont}(T) = (m_1, m_2, \ldots)$ is defined by $m_i := |\{\text{cells in } T \text{ with entry } i\}|$ for all $i \ge 0$.

We shall also use the Frobenius notation for partitions, defined as follows: Let λ be a partition of n and set $d := \max\{i \mid \lambda_i - i \ge 0\}$ (i.e., the length of the main diagonal in the Young diagram of λ). Then the Frobenius notation for λ is $(\lambda_1 - 1, \ldots, \lambda_d - d \mid \lambda'_1 - 1, \ldots, \lambda'_d - d)$.

For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n define the following doubling operation

$$2 \cdot \lambda := (2\lambda_1, \ldots, 2\lambda_k) \vdash 2n.$$

If all the parts of λ are distinct, define also

$$2 * \lambda := (\lambda_1, \ldots \lambda_k \mid \lambda_1 - 1, \ldots \lambda_k - 1) \vdash 2n$$

in the Frobenius notation.

3. The Littlewood-Richardson Coefficients

Let $\bar{a} = (a_1, a_2, \ldots, a_n)$ be a sequence of positive integers. $\bar{a}$ is called a *reverse ballot sequence* if for every $1 \le i < n$ and $1 \le j \le n$ the number of occurences of i in the prefix $(a_1, \ldots, a_j)$ is not less than the number of occurences of $i+1$ in $(a_1, \ldots, a_j)$.

A semistandard Young tableau of shape λ/μ is *proper* if, when reading its entries from right to left, starting in the topmost row and going down, we obtain a reverse ballot sequence.

The Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is the number of proper semistandard Young tableaux of shape λ/μ and content vector ν.

The irreducible S_n-modules (Specht modules) will be denoted by S^{λ}, and the irreducible $GL_k(\mathbb{C})$-modules (Weyl modules) by V_k^{λ}. The Littlewood-Richardson coefficients describe the decomposition of tensor products of Weyl modules. Let $\mu \vdash t$ and $\nu \vdash n-t$. Then

$$V_k^{\mu} \otimes V_k^{\nu} \cong \bigoplus_{\lambda \vdash n} c^{\lambda}_{\mu\nu} V_k^{\lambda},$$

for $k \ge \max\{\ell(\lambda), \ell(\mu), \ell(\nu)\}$ (and the coefficients $c^{\lambda}_{\mu\nu}$ are then independent of k).

By Schur-Weyl duality they are also the coefficients of the outer product of Specht modules. Namely,

$$(S^{\mu} \otimes S^{\nu}) \uparrow_{S_t \times S_{n-t}}^{S_n} \cong \bigoplus_{\lambda \vdash n} c^{\lambda}_{\mu\nu} S^{\lambda}.$$

Let λ and μ be two partitions of the same integer n, and let $0 \le i \le n$. Define

$$c^{\lambda\mu}(i) := \sum_{\alpha\vdash n-i,\ \beta\vdash i} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\beta'} \ .$$

Thus $c^{\lambda\mu}(i)$ is the number of pairs of proper semistandard Young tableaux of shapes λ/α, μ/α respectively (where α is some partition of $n-i$) with conjugate content vectors.
Example.

$$c^{\lambda\mu}(0) = \delta_{\lambda\mu} \quad , \quad c^{\lambda\mu}(n) = \delta_{\lambda\mu'} \ . \tag{3.1}$$

We shall use also the following notation for *extended Littlewood-Richardson coefficients*:

$$c^{\lambda}_{\alpha\beta\gamma\delta} := \sum_{\mu,\nu} c^{\lambda}_{\alpha\mu} c^{\mu}_{\beta\nu} c^{\nu}_{\gamma\delta};$$

so that

$$V^{\alpha}_{k} \otimes V^{\beta}_{k} \otimes V^{\gamma}_{k} \otimes V^{\delta}_{k} = \bigoplus_{\lambda} c^{\lambda}_{\alpha\beta\gamma\delta} V^{\lambda}_{k}.$$

4. Symmetric and Exterior Powers of Matrix Spaces

In this section we cite well-known classical theorems, concerning the decomposition into irreducibles of symmetric and exterior powers of matrix spaces, which are to be generalized in this paper.

Let $M_{k,m}$ be the vector space of $k\times m$ matrices over $\mathbb{C}$. Then $M_{k,m}$ carries a (left) $GL_k(\mathbb{C})$-action and a (right) $GL_m(\mathbb{C})$-action. A classical Theorem of Ehresmann [3] (see also [11]) describes the decomposition of an exterior power of $M_{k,m}$ into irreducible $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$-modules.

Theorem 4.1. *The n-th exterior power of $M_{k,m}$ is isomorphic, as a $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$-module, to*

$$\wedge^n(M_{k,m}) \cong \bigoplus_{\lambda\vdash n \text{ and } \lambda\subseteq(m^k)} V^{\lambda}_{k} \otimes V^{\lambda'}_{m},$$

where λ' is the partition conjugate to λ.

The following three results on symmetric powers were proved several times independently; these results may be found in [7] and [4].
The symmetric analogue of Theorem 4.1 was studied, for example, in [7, (11.1.1)] and [4, Theorem 5.2.7].

Theorem 4.2. *The n-th symmetric power of $M_{k,m}$ is isomorphic, as a $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$-module, to*

$$Sym^n(M_{k,m}) \cong \bigoplus_{\lambda\vdash n \text{ and } \ell(\lambda)\le\min(k,m)} V^{\lambda}_{k} \otimes V^{\lambda}_{m}.$$

Let $M^{+}_{k,k}$ be the vector space of symmetric $k \times k$ matrices over $\mathbb{C}$. This space carries a natural two sided $GL_k(\mathbb{C})$-action. The following theorem describes the decomposition of its symmetric powers into irreducible $GL_k(\mathbb{C})$-modules.

Theorem 4.3. *The n-th symmetric power of $M^{+}_{k,k}$ is isomorphic, as a $GL_k(\mathbb{C})$-module, to*

$$Sym^n(M^{+}_{k,k}) \cong \bigoplus_{\lambda\in Par_k(n)} V^{2\cdot\lambda}_{k}.$$

This theorem was proved by A.T. James [8], but had already appeared in an early work of Thrall [19]. See also [6], [15], [7, (11.2.2)] and [4, Theorem 5.2.9] for further proofs and references.

Let $M^-_{k,k}$ be the vector space of skew symmetric $k \times k$ matrices over $\mathbb{C}$. Then

Theorem 4.4. *The n-th symmetric power of $M^-_{k,k}$ is isomorphic, as a $GL_k(\mathbb{C})$-module, to*

$$Sym^n(M^-_{k,k}) \cong \bigoplus_{(2\cdot\lambda)' \in Par_k(2n)} V_k^{(2\cdot\lambda)'}.$$

This theorem was proved in [5], [6], [15]. See also [7, (11.3.2)] and [4, Theorem 5.2.11].

5. Main Results

Let $M_{k,m}$ be the vector space of $k \times m$ matrices over $\mathbb{C}$. The tensor power $M_{k,m}^{\otimes n}$ carries a natural S_n-action by permuting the factors. This action decomposes the tensor power into irreducible S_n-modules. Let $M_{k,m}^{\otimes n}(i)$ be the isotypic component of $M_{k,m}^{\otimes n}$ corresponding to the irreducible S_n-representation indexed by the hook $(n-i, 1^i)$, where $0 \le i \le n-1$. This component still carries a $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$-action. Its decomposition into irreducibles is given by a convolution of the Littlewood-Richardson coefficients.

Theorem 5.1. *Let λ and μ be partitions of n, of lengths at most k and m, respectively. For every $0 \le i \le n$ the multiplicity of the irreducible $GL_k(\mathbb{C}) \times GL_m(\mathbb{C})$-module $V_k^\lambda \otimes V_m^\mu$ in $M_{k,m}^{\otimes n}(i-1) \oplus M_{k,m}^{\otimes n}(i)$ is the restricted convolution $c^{\lambda\mu}(i)$, as defined in Section 3 above. By convention, $M_{k,m}^{\otimes n}(-1) = M_{k,m}^{\otimes n}(n) = 0$.*

Theorem 5.1 interpolates between two well-known classical theorems, Theorems 4.1 and 4.2. Indeed, $M_{k,m}^{\otimes n}(0) \cong \mathrm{Sym}^n(M_{k,m})$ and $M_{k,m}^{\otimes n}(-1) = 0$. Substituting $i = 0$ and applying (3.1) shows that the relevant multiplicity is $\delta_{\lambda\mu}$, thus proving Theorem 4.2. Similarly, the substitution $i = n$ gives Theorem 4.1.

The following corollary generalizes the duality between Theorem 4.1 and Theorem 4.2.

Corollary 5.2. *Let $\mu \subseteq (m^m)$ and λ be partitions of n. For every $0 \le i \le n-1$ the multiplicity of $V_k^\lambda \otimes V_m^\mu$ in $M_{k,m}^{\otimes n}(i)$ is equal to the multiplicity of $V_k^\lambda \otimes V_m^{\mu'}$ in $M_{k,m}^{\otimes n}(n-1-i)$.*

Let λ and μ be partitions of n. Define the *distance*

$$d(\lambda, \mu) := \frac{1}{2} \sum_i |\lambda_i - \mu_i|.$$

Theorem 5.1 together with results of Regev [14, Theorem 12] and Dvir [2, Theorem 1.6] imply

Theorem 5.3. *If $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M_{k,m}^{\otimes n}(t)$ (for some $0 \le t \le n-1$) then*

$$d(\lambda, \mu) < km.$$

This shows that, for $V_k^\lambda \otimes V_m^\mu$ to appear in a hook component, λ and μ must be very "close" to each other (for k and m fixed, n tending to infinity).

Consider now the vector space $M_{k,k}$ of $k \times k$ square matrices over $\mathbb{C}$. Let $M_{k,k}^{\otimes n}(i,j)$ be the component of $M_{k,k}^{\otimes n}(i)$ consisting of tensors with j skew symmetric and $n-j$ symmetric factors. $M_{k,k}^{\otimes n}(i,j)$ carries a $GL_k(\mathbb{C})$ two-sided diagonal action. The following theorem describes its decomposition as a $GL_k(\mathbb{C})$-module.

Theorem 5.4. *Let λ be a partition of $2n$ of length at most k. For every $0 \le i \le n$ and $0 \le j \le n$ the multiplicity of V_k^λ in $M_{k,k}^{\otimes n}(i,j) \oplus M_{k,k}^{\otimes n}(i-1,j)$ is*

$$\sum_{|\alpha|+|\beta|+|\gamma|+|\delta|=n,\ |\beta|+|\delta|=j\ ,\ |\gamma|+|\delta|=i} c^{\lambda}_{2\cdot\alpha,(2\cdot\beta)',2*\gamma,(2*\delta)'},$$

where the sum runs over all partitions $\alpha, \beta, \gamma, \delta$ with total size n such that γ and δ have distinct parts, β and δ have total size j, and γ and δ have total size i. The operations $*$ and $\cdot$ are as defined in Section 2, and the extended Littlewood-Richardson coefficients $c^\lambda_{\alpha\beta\gamma\delta}$ are as defined in Section 3.

The proof of Theorem 5.4 involves results on plethysm of elementary and homogeneous symmetric functions [13, Ch. I §8 Ex 5-6].

Theorem 5.4, for $i = 0$, interpolates between classical results, regarding symmetric powers of the spaces of symmetric and skew symmetric matrices (Theorems 4.3 and 4.4). Another boundary case, $i = n$, gives an interpolation between exterior powers of the same matrix spaces.

Corollary 5.5. *Let $\lambda \subseteq (k^k)$ be a partition of $2n$. For every $0 \le i \le n-1$ and $0 \le j \le n$, the multiplicity of V_k^λ in $M_{k,k}^{\otimes n}(i,j)$ is equal to the multiplicity of $V_k^{\lambda'}$ in $M_{k,k}^{\otimes n}(i, n-j)$.*

For proofs and more details see [1].

Acknowledgments. The authors thank R. Howe and N. Wallach for their useful comments.

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Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel

E-mail address: radin@math.biu.ac.il

Sackler School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel

E-mail address: frumkin@math.tau.ac.il

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel

E-mail address: yuvalr@math.biu.ac.il